# Swimming microbots: Dissipation, optimal stroke and scaling

J.E. Avron[1], O. Gat[2], and O. Kenneth[1]

[1] Department of Physics, Technion, Haifa 32000, Israel

[2] Department of Electrical Engineering, Technion, Haifa, 32000, Israel

March 25, 2004

**Abstract**

Micron size robots which swim at macroscopic speeds need much more power for locomotion than organisms of similar size. This makes power optimization important. We formulate a notion of optimal swimming which allows us to find optimal strokes. A central role is played by a dimensionless quality factor, *the swimming drag coefficient*, which we introduce. This allows us to derive general scaling relations, showing for example that the maximal speed of an externally powered robot is proportional to the square root of its length. We test these ideas on a soluble model of a two dimensional swimmer where the optimal stroke is determined explicitly.

## 1 Introduction

Micron size autonomous devices that navigate and perform useful tasks make one of the grand visions of robotics. The micron scale is sufficiently large to accommodate complex internal structures—a prerequisite to an autonomous smart device—and at the same time, is small enough to interface with functional microscopic biological systems. Although, at present, we are still quite far from building swimming robots on this scale [1], currents efforts in nanomechanics and micro-machining [2] justify looking into theoretical questions associated with the locomotion of micron size robots that need to traverse macroscopic distances at reasonable time scales.

Hydrodynamics at the microscopic and macroscopic scales are different because inertia becomes negligible at small scales [4, 5]. As a consequence, micron size robots would most likely have to rely on propulsion mechanisms that differ from their macroscopic analogs. At the same time they can not quite imitate the swimming of microscopic organisms, such as bacteria, either, even though their length scales are comparable. This is because bacteria move slowly. Robots should swim much faster if they are to interface with the macroscopic world. The disparity in velocity has dramatic consequences for the power consumption: A microbot swimming 100 times as fast as a bacterium, at the modest speed of 1 mm per second, would consume $10^4$ more power. This means that microbots should attempt to swim as effectively as they can while organisms may prefer to optimize other energy consuming functions.

Many microscopic organisms swim by small squirming or undulating strokes [4, 6, 7, 8]. Optimal swimming with small strokes has been studied, in [9, 8]. However, once large strokes are allowed, one sees that small strokes can not be optimal. Efficient swimming involve strokes that are likely to be on the same scale as the swimmer itself.

To be of practical value, the optimal stroke must be independent of the distance traveled. This means that doubling the distance comes from doubling the number of strokes and not from changing the stroke. We call this upward scaling. One may imagine a small object ejecting a slender pseudopod so as to swim a distance that is much larger than its own size in a single stroke. One of our aims is to show that this scenario can sometimes be ruled out without appeal to specific material properties.

We study a soluble model of a micro-swimmer in two dimensions and determine its optimal stroke (in a sense discussed below). The stroke has the requisite upward scaling property and is, as one might expect, of the same scale as the swimmer itself. It is qualitatively different from the small strokes of squirming bacteria, and the undulating strokes of nematodes.

The problem of optimal swimming can be put in a wider context of control theory: The shape of the swimmer is the control, the displacement is the output and the dissipation is the cost function that needs to be minimized subject to appropriate constraints. For related applications in control theory see [11] and references therein. We note that swimming does not necessarily require an ambient fluid: J. Wisdom [14] recently pointed out that, remarkably, it is also possible to swim in empty space provided space time is curved. Deformable stars can, in principle, swim near the horizon of a black hole.

## 2 Swimming is geometric [10]

Microbots, like microorganisms, live in a world dominated by viscosity. This fact is expressed by the Reynolds number, $Re = \rho UL/\eta$, being small. $\rho$ is the fluid density, $\eta$ the viscosity, $U$ the velocity and $L$ the scale of the object. A bacterium that moves at speeds of say ten body lengths per second has Reynolds number of the order of $10^{-5}$ in air or water [4]. Microbots that swim 100 faster than similar size bacteria, have $Re$ of the order of $10^{-3}$, which is still conveniently small.

Since inertia can be neglected at low Reynolds numbers the total force and torque on the swimmer must vanish. For Stokes flows, (i.e. a solution of Navier-Stokes equations without the inertial terms) this gives a linear relation between infinitesimal displacement of the swimmer to the infinitesimal changes in its shape. Swimming at low Reynolds numbers may therefore be interpreted, somewhat intriguingly, as an expression of the fact that the fluid *does not* generate net forces on the swimmer.

A stroke is a closed parameterized path $\gamma$ in shape space. We call the displacement due to one stroke the step size of the swimmer. Since the swimmer moves by deformations, different physical points in the swimmer follow different trajectories during the stroke. However, after a full stroke, all points are displaced by the same step. To calculate the step, one may fix a fiducial point in the body and study its change of position due to an (infinitesimal) change in the controls. This (linear) relation is known geometrically as a *connection* on the space of controls [1]. The displacement in one stroke expresses the fact that the position of the swimmer *is not* a function on shape space; there is anholonomy that makes the actual position path dependent. In this respect, the displacement in swimming is a classical analog of the Berry's phase [12] which expresses the anholonomy of the quantum mechanical phase under adiabatic evolutions.

A consequence of the geometric description is the well known fact that no swimming is possible at low Reynolds numbers for strokes that are self-retracing (reciprocal), a fact known as the scallop theorem [5]. Another general consequence is that, for small strokes, the displacement scales like the area enclosed by $\gamma$. This follows from a simple application of Stokes' theorem which converts a line integral over the closed path in shape space to

[1] The vector potential of electrodynamics is a prototype of a connection and the gauge freedom is the analog of the freedom to choose a fiducial point.

area integral. As we shall see in the next section this renders small strokes ineffective.

# 3 The swimming drag coefficient

Minimizing the drag a is central objective in many engineering problems including the design of planes, cars and buildings. The drag at high Reynolds numbers of an object with a fixed shape is characterized by the drag coefficient, $C_D$, which is the dimensionless number defined by [13]

$$C_D = \frac{2\,F_D}{\rho U^2 L^{d-1}} \ . \tag{1}$$

$F_D$ is the drag force and $d$ the number of space dimensions.

There are several difficulties in trying to adapt this notion to swimming at low Reynolds numbers. The first is that $C_D$ diverges like $1/Re$ as $Re$ tends to zero. This can be seen, for example, by letting $U \to 0$ and recalling that in Stokes flows $F_D$ is proportional to $U$. The divergence suggests that the drag for Stokes flow needs to be renormalized by a factor of $Re$.

Another complication is that, unlike planes or cars, swimmers do not have a fixed shape. What, then, should one use for $U$ and $L$ and what does one mean by $F_D$? Different parts of the swimmer have different velocities during the stroke and there is not natural way to separate drag from thrust in a swimmer.

We resolved these difficulties by introducing the *swimming drag coefficient* defined by

$$\delta(\gamma) = \frac{D(\gamma)\tau(\gamma)}{\eta X^2(\gamma) L^{d-2}} \ . \tag{2}$$

$D(\gamma)$ is the energy dissipated in a stroke $\gamma$ and $X(\gamma)$ the step size. $\tau(\gamma)$ is the period of the stroke. Unlike $C_D$ which is just a number for an object of fixed shape moving at constant speed, $\delta(\gamma)$ is a functional on the space of strokes (parameterized closed paths). $\delta(\gamma)$ is formally equivalent to $Re \cdot C_D$ once one identifies $F_D(\gamma) = D(\gamma)/X(\gamma)$ and $U(\gamma) = X(\gamma)/\tau(\gamma)$.

Since swimming is geometric $X(\gamma)$ is independent of the parametrization of the stroke $\gamma$. $D(\gamma)$, on the other hand, depends on the parametrization and is inversely proportional to the period. (The power in Stokes flows is proportional to the velocity squared.) The product $D(\gamma)\tau(\gamma)$ is, however, independent of the period of the stroke.

Optimal strokes are those that minimize $\delta(\gamma)$. Minimizing $\delta$ is equivalent to minimizing the energy dissipated in swimming a given distance, *at a given average speed*.

In two dimensions the optimal stroke is independent of the scale $L$ of the swimmer. In three dimensions smaller swimmers are more efficient in the sense that they swim a given distance in a given time with less dissipation than geometrically similar larger swimmers.

Small strokes make $\delta(\gamma)$ large and are therefore inefficient. For small strokes, both $D(\gamma)$ and $X(\gamma)$ scale with the area enclosed by the stroke. Hence, $\delta(\gamma)$ diverges like the inverse area as the stroke $\gamma$ gets smaller.

It is instructive to contrast $\delta$ with a different notion of efficiency introduced by Shapere and Wilczek in their study of squirmers [9]. Since Shapere and Wilczek did not formulate their criterium in terms of a dimensionless quantity, we first need to rewrite their notion in the corresponding dimensionless form. Shapere and Wilczek require that the optimal stroke minimizes

$$\epsilon(\gamma) = \frac{D(\gamma)\tau}{\eta X(\gamma) L^{d-1}} \,. \tag{3}$$

The difference between $\epsilon$ and $\delta$ is that one power of $X$ was traded for one power of $L$. $\epsilon$ optimizes the dissipation per unit length, *but without a constraint of fixed velocity*. The motivation for the definition of $\epsilon(\gamma)$ comes from its scale invariance for small strokes. (Since both the dissipation and the step scale with the area of $\gamma$.) The optimal stroke determined by $\epsilon$ is therefore fixed only up to an overall scale factor. This is a desirable property when one discusses the swimming of ciliated bacteria where the amplitude of the stroke is constrained by the small size of the cilia. In the absence of a constraint on the amplitude the scale should emerge from the optimization and $\epsilon$ must be replaced by $\delta$.

# 4 A model swimmer in two dimensions

Shapere and Wilczek [10] introduced a soluble model of a two dimensional swimmer. We shall use their model to test the hypothesis that minimizing $\delta$ leads to an optimal finite stroke with good upward scaling.

In this model the shape of the swimmer at time $t$ is given as the image of units circle, $|\zeta| = 1$, under the Riemann map

$$z(\zeta;t) = W(t)\zeta + X(t) + \frac{Y(t)}{\zeta} + \frac{Z(t)}{\sqrt{2}\,\zeta^2}\,. \tag{4}$$

The real and imaginary parts of $z$ give the $x$ and $y$ coordinates of the boundary of the swimmer.

When $Z = 0$ and $|W| \neq |Y|$ the shape described by Eq. (4) is a (shifted) ellipse. By a symmetry argument an ellipse can rotate but not swim. In this sense the model, Eq. (4), with non-zero $Z$ gives a minimal example of a swimmer.

Wilczek and Shapere allowed $W(t), X(t), Y(t)$ and $Z(t)$ to be complex valued which allowed them to consider both translations and rotations. We shall restrict $\{W(t), Y(t), Z(t)\}$ to be real. Shape space is then three dimensional and describes shapes that are symmetric under mirror reflection. (This follows from $\bar{z}(\zeta;t) - X = z(\bar{\zeta};t) - X$ where $\bar{z}$ denote the complex conjugate of $z$.) $X(t)$ is the (fiducial) position parameter. A reflection symmetric swimmer can not rotate and can only swim in the x-direction. Hence, without loss, $X(t)$ may be taken to be real as well.

The flow outside the swimmer, $|\zeta| \geq 1$, is given by [10]:

$$v = f_1(\zeta) + \overline{f_2(\zeta)} - z(\zeta)\,\overline{\left(\frac{f_1'(\zeta)}{z'(\zeta)}\right)}, \quad v = v_x + iv_y\,, \tag{5}$$

with

$$f_1 = \frac{\dot{Y}}{\zeta} + \frac{\dot{Z}}{\sqrt{2}\zeta^2},\ f_2 = \dot{X} - \frac{\frac{W}{\zeta} + X + Y\zeta + \frac{Z\zeta^2}{\sqrt{2}}}{W - \frac{Y}{\zeta^2} - \frac{\sqrt{2}\,Z}{\zeta^3}}\left(\frac{\dot{Y}}{\zeta^2} + \frac{\dot{Z}\sqrt{2}}{\zeta^3}\right) + \frac{\dot{W}}{\zeta}\,. \tag{6}$$

Where $\dot{Y}$ denotes the time derivative of $Y(t)$. The requirement that the flow vanishes at infinity gives the requisite relation between the response, $đX$, and the control

$$đX = A\,dY, \qquad A = \frac{Z}{\sqrt{2}W}\,, \tag{7}$$

where the notation $đX$ is used to stress that the form is not exact, it does not integrate to a function on the space of shapes. Geometrically, this relation is interpreted as the connection on the space of shapes [9].

The power $P$ dissipated by the swimmer is calculated by integrating the stress times the velocity on the surface of the swimmer:

$$P = \mathrm{Im} \oint \bar{v}(\eta \bar{\partial} v d\bar{z} - p dz) \ , \tag{8}$$

where the integration is on the swimmer boundary, $|\zeta| = 1$, and $p$ is the fluid pressure, $p = -4\eta \, \mathrm{Re} \, f_1'(z)$. Using the no-slip boundary conditions and the explicit expression, Eq. (6), one obtains

$$P = 4\pi\eta \left( \dot{W}^2 + \dot{Y}^2 + \dot{Z}^2 \right) . \tag{9}$$

The dissipation of a stroke, $D(\gamma)$, is then

$$D(\gamma) = 4\pi\eta \int_0^\tau \left( \dot{W}^2 + \dot{Y}^2 + \dot{Z}^2 \right) dt \ . \tag{10}$$

Since the integrand in Eq. (10) is the Euclidean length element squared (in shape space), we can use well-known properties of such integrals to find the optimal parametrization of $\gamma$ which minimized $D(\gamma)$. Namely, $D(\gamma)$ is minimized when $\gamma$ is traversed at constant speed, where it takes the value

$$D_{\min}(\gamma) = 4\pi\eta \frac{|\gamma|^2}{\tau} \ . \tag{11}$$

$|\gamma|$ is the Euclidean length of $\gamma$. Plugging this in Eq. (1) we find that minimizing the drag coefficient $\delta$ is equivalent to minimization of the ratio of the stroke length in shape space to the step size in coordinate space, $|\gamma|/X(\gamma)$.

# 5 Physical constraints

A stroke is a closed path $\gamma$ in the $\{W, Y, Z\}$ space. However, not all strokes are physical. One restriction comes from the fact that a physical shape can not self-intersect. The other restriction is that a swimmer is incompressible and hence must preserve its volume (area in our case).

We call a shape *simple* if it does not self intersect. When $Z = 0$ and $W > |Y|$ the shapes described by Eq. (4) are ellipses, hence simple. (The ellipse degenerates to a line segment when $W = \pm Y$).

Let $C$ denote the region in shape space corresponding to simple shapes that are deformations of the circle $W = 1, Y = Z = 0$. $C$ is evidently a cone in the space $\{W, Y, Z\}$. The boundary of the cone consists of points $\{W, Y, Z\}$ for which $z'(\zeta) = 0$ for some $\zeta$ of unit modulus. A short computation shows that the cone is bounded by two planes:

$$W = Y \pm \sqrt{2}\, Z\,, \tag{12}$$

and the parabolic cone:

$$WY = 2Z^2 - W^2\ . \tag{13}$$

The intersection of the cone of simple shapes with the plane $W = 1$ is shown in Fig. 1. The shapes inside the cone $C$ are smooth. On the boundary

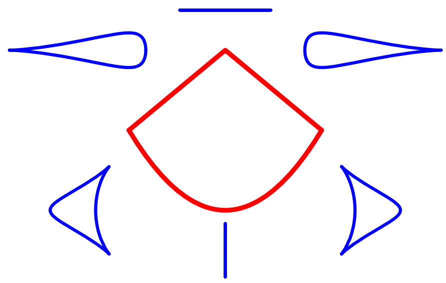

Figure 1: *The section of the cone of simple shapes, $C$, with the plane $W = 1$ (red). The center of $C$ corresponds to a circular shape and the vertical axis to ellipses. Inside $C$ the shape is smooth. On the boundary of $C$ the shape develops cusps. Six shapes (blue, thin curves) are drawn for representative points on the boundary. The shape degenerates to a line at two points on the boundary.*

the shapes are not smooth and have cusp-like singularities which are the precursor of self-intersections.

Let us now turn to swimmers which preserve their area while swimming. The area of the swimmer whose shape is given by Eq. (4) is

$$\frac{1}{2}\,\mathrm{Im}\oint \bar{z}\,dz = \pi(W^2 - Y^2 - Z^2)\,. \tag{14}$$

Fixing the area corresponds to restricting the stroke to a hyperboloid in shape space. We may choose the unit of area so that the area of the swimmer is $\pi$.

The pagoda-like domain of Fig. 2 delineates the $(Y, Z)$ coordinates of the domain of physical shapes. The $W$ coordinate is then fixed by Eq.(14). Physical strokes are represented by closed paths that lie inside the "pagoda" of Fig. 2.

# 6 Large strokes are inefficient

The domain of physical shapes, which we denote by $P$, is shaped like a "pagoda" which extends vertically without a limit. This allows for swimming large distances with a single stroke. (The slender pseudopod scenario.) However, as we shall now show, this mode of swimming is inefficient.

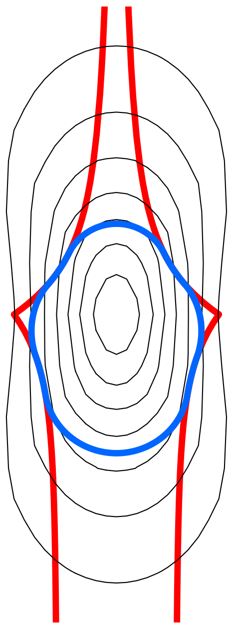

Figure 2: *The pagoda-like curve (red) is the boundary of the intersection of the cone of simple shapes with the hyperboloid of unit area projected on the $Y$-$Z$ plane. The asymptotic width of the strip going to $Y = -\infty$ is $\Delta Z = 2/\sqrt{3}$. The inscribed blue curve is the image of the optimal stroke. The (black) contours show the level sets of the magnetic field which decreases fast in the $Z$ direction but slowly in the $Y$ direction.*

It follows from Eq. (7) that

$$X(\gamma) = \oint_\gamma A\, dY \tag{15}$$

The domain of simple, incompressible shapes is contained in the strip $|Z| \leq 1$. Therefore, since $A = O(\frac{1}{Y})$, a long excursion, of order $\ell$, in the $Y$ direction contributes $O(\log \ell)$ to $X(\gamma)$, but $O(\ell)$ to $|\gamma|$. As $\ell \to \infty$ the drag $\delta$ diverges like $\ell/\log\ell$. Hence, the slender pseudopod scenario is inefficient. Since the drag also diverges for small strokes, it follows that the minimizer of $\delta$ is a finite stroke. Swimming large distances is best achieved by repeating the optimal, finite stroke many times.

# 7 Optimal stroke: Quantitative aspects

We shall now describe how one can calculate the optimal stroke. As we shall show the problem reduces to studying orbits of a charged particle in a magnetic field.

We first solve an auxiliary problem which is of independent interest, namely, to find the stroke which minimizes the dissipation for a *given* step $X(\gamma)$. This stroke is the minimizer of the functional

$$S_q[\gamma] = 4\pi\eta \int_0^\tau \big(\dot W^2 + \dot Y^2 + \dot Z^2\big)dt + q\int_0^\tau A\dot Y\,dt, \qquad \gamma \in P\,, \tag{16}$$

where $q$ is a Lagrange multiplier. $S_q$ can be interpreted as the action of a particle of charge $q$ in a gauge field with the vector potential $A\hat Y$. By general principles the minimizing $\gamma$ either lies on the boundary of the domain, $\partial P$, or must solve the Euler-Lagrange equations, i.e., must be an orbit of a charged particle in the magnetic field $B = \nabla\times(A\hat Y)$. Since a magnetic field does no work on the particle, the velocity is constant as was already observed.

The minimizer of $S_q$ is a continuously differentiable curve. For suppose that the curve $\gamma$ has a corner, cutting it at a distance $\varepsilon$ decreases $|X(\gamma)|$ by $\varepsilon^2$. At the same time, the length $\gamma$ decreases by $\varepsilon$. This implies that $\gamma$ can not have corners.

Since the minimizing $\gamma$ is smooth, it must avoid the corners of the boundaries, $\partial P$. Moreover, when it leaves the boundary it must do so tangentially. Using these observations the orbit, $\gamma_q$, for any $q$, can be computed numerically. The optimal stroke is the one that minimizes the ratio $|\gamma_q|/X(\gamma_q)$, which is a minimization problem in one variable. The curve of the optimal stroke is shown in fig. 2 while snapshots of the optimal stroke are shown in Fig. 3. A movie of the swimmer can be viewed in [15]. For the optimal stroke we find that the stroke to step size is $|\gamma|/X(\gamma) \approx 3.02\,$,. The corresponding swimming drag coefficient is $\delta_{\text{optimal}} \approx 114$ .

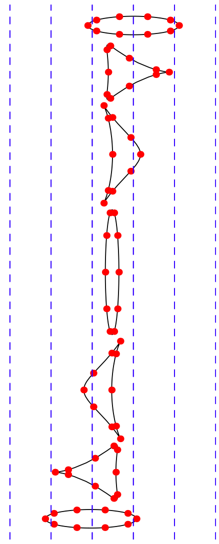

Figure 3: Snapshots of the optimal stroke and motion of a model swimmer. The snapshots are shifted vertically for visibility. The first and last snapshots are related by a translation. The shapes with cusps correspond to those parts of the stroke that lie on the boundary of the domain of of simple shapes. The (red) dots are points that are fixed in the body.

# 8 Perspectives

The locomotion of microbots raises certain theoretical questions that can be treated independently of material properties. These have been addressed here. We have introduced the notion of *swimming drag coefficient* and showed that minimizing the drag coefficient leads to optimal swimmers. In the model of Wilczek and Shapere one can rule out the slender pseudopod scenario of swimming large distances with few strokes and we found the optimizing stroke which admits upward scaling.

It follows from the definition of the swimming drag coefficient that the power needed to swim in three dimensions is $P = \delta\eta U^2 L$. Although this says that smaller objects need less power for locomotion, the available power usually also scales as some power of $L$. For a swimmer with external power source in the form of flux of radiation, the available power scales like $L^2$. This leads to maximal speed, $U$, that scales with $\sqrt{L}$. For a swimmer with an internal energy source, the available total energy scales like $L^3$ and the maximal distance it can swim then scales like $L^2/U$. In either case, there is an advantage to size (provided distances are measured in absolute units).

Some intuition into the swimming at low Reynolds numbers can be gained by inspecting Fig. 3: When the swimmer is approximately triangular, the base of the triangle functions as an anchor that pushes or pulls the opposite vertex. The swimming strategy is then to interpolate between right-pointing and left-pointing triangles through circular shapes. Much of the actual push is then done by the triangles.

Linear deformations of polygons are attractive in an engineering context. Unfortunately, they do not give rise to simple soluble models but can be studies numerically. It would be interesting to find the swimming drag coefficients of few simple polygons.

It is likely that the optimal swimmer in two dimensions is not the optimal swimmer of the Wilczek and Shapere model that we found here. However, its swimming drag coefficient of $\delta \approx 114$ gives a benchmark that better swimmers would have to beat. It would be interesting to compare it with the drag coefficient of other swimming styles, e.g. the waving of flagella.

Although it is hard to compute the swimming drag coefficient, even for idealized models in two dimensions, it is, in principle at least, an easily measurable quantity. Using the similarity rules of hydrodynamics [13] one can measure the coefficient of drag for *scaled up* models of the swimmer and search experimentally for smaller swimming drag coefficient and improved strokes.

We thank E. Braun , G. Kosa, M. Shoham and D. Weihs and for useful discussions, and U. Sivan for proposing the problem and critical reading of the manuscript. This work is supported in part by the EU grant HPRN-CT-2002-00277.

# References

[1] Fukuda T.et al.,(1995), *Steering Mechanism and Swimming Experiment of Micro Mobile Robot in Water*, Proc. of Micro Electro Mechanical Systems (MEMS'95),300-305; Guo S. et. al.,(2002), *A Fin Type of Microrobot in Pipe*, Proc.of the 2002 Int'l Symposium on Micromechatronics and Human Science (MHS 2002), 93-98; G. Kosa, Research porposal, (2002), Technion, http://robotics.technion.ac.il/

[2] M. Frederick Hawthorne, Jeffrey I. Zink, Johnny M. Skelton, Michael J. Bayer, Chris Liu, Ester Livshits, Roi Baer, Daniel Neuhauser, *Electrical or Photocontrol of the Rotary Motion of a Metallacarborane*, Science, **303**, 1849 (2004).

[3] R. Hollis, *Whither microbots?*, Proc. 7th Int'l Conf. on Micromachines and Human science, Nagoya, (1996) and references therein.

[4] H. C. Berg, *Random walks in Biology*, Princeton, (1983)

[5] E.M. Purcell, *Life at low Reynolds numbers*, Am. J. Physics **45**, 3-11 (1977).

[6] W. Ludwig, Zeit. f. Vergl. Physiolg. **13**, 397-503, (1930); J. Lighthill, *On the squirming motion of nearlt spherical deformable bodies through liquids at very small Reynolds numbers*, Comm. Pure. App. Math. **5**, 109-118 (1952); J.R. Blake, *A spherical envelops approach to ciliary propulsion*, J. Fluid. Mech. **46**, 199-208 (1971)

[7] K. M. Ehlers, A.D. Samuel, H.C. Berg and R. Montgomery, *Do Cyanobacteria swim using traveling surface waves*, PNAS **93**, 8340-8343 (1996).

[8] J. Koiller, R. Montgomery and K. Ehlers, *Problems and progress in microswimming*, J. Nonliear Sci. **6**, 507-541 (1996); J, Koiller, M. Raup, M. Delgado, J. Ehlers and K.M. Montgomery, Comm. App. Math **17** 3 (1998). H.A. Stone and A.D. Samuel, *Propulsion of micro-organisms by surface distortions*, Phys. Rev. Lett. **77**, 4102-4104 (1996).

[9] A. Shapere and F. Wilczek, *Efficiency of self-propulsion at low Reynolds numbers*, J. Fluid Mech., **198**, 587-599 (1989).

[10] A. Shapere and F. Wilczek, *Geometry of self-propulsion at low Reynolds numbers*, J. Fluid Mech., **198**, 557-585 (1989).

[11] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden and R.M. Murray, *Non holonomic mechnaical systems with symmetry*, Arch. Rational mech, Anal. **136**, 21-99 (1996).

[12] M.V. Berry, *Quantal phase factors accompanying adiabatic changes*, Proc. Roy. Soc. London A **392**, 45 (1984).

[13] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, Pergamon (1959)

[14] J. Wisdom, *Swimming in curved space time: Motion by cyclic changes in body shape*, Science, **299**, 1865-1869 (2003).

[15] http://physics.technion.ac.il/~avron