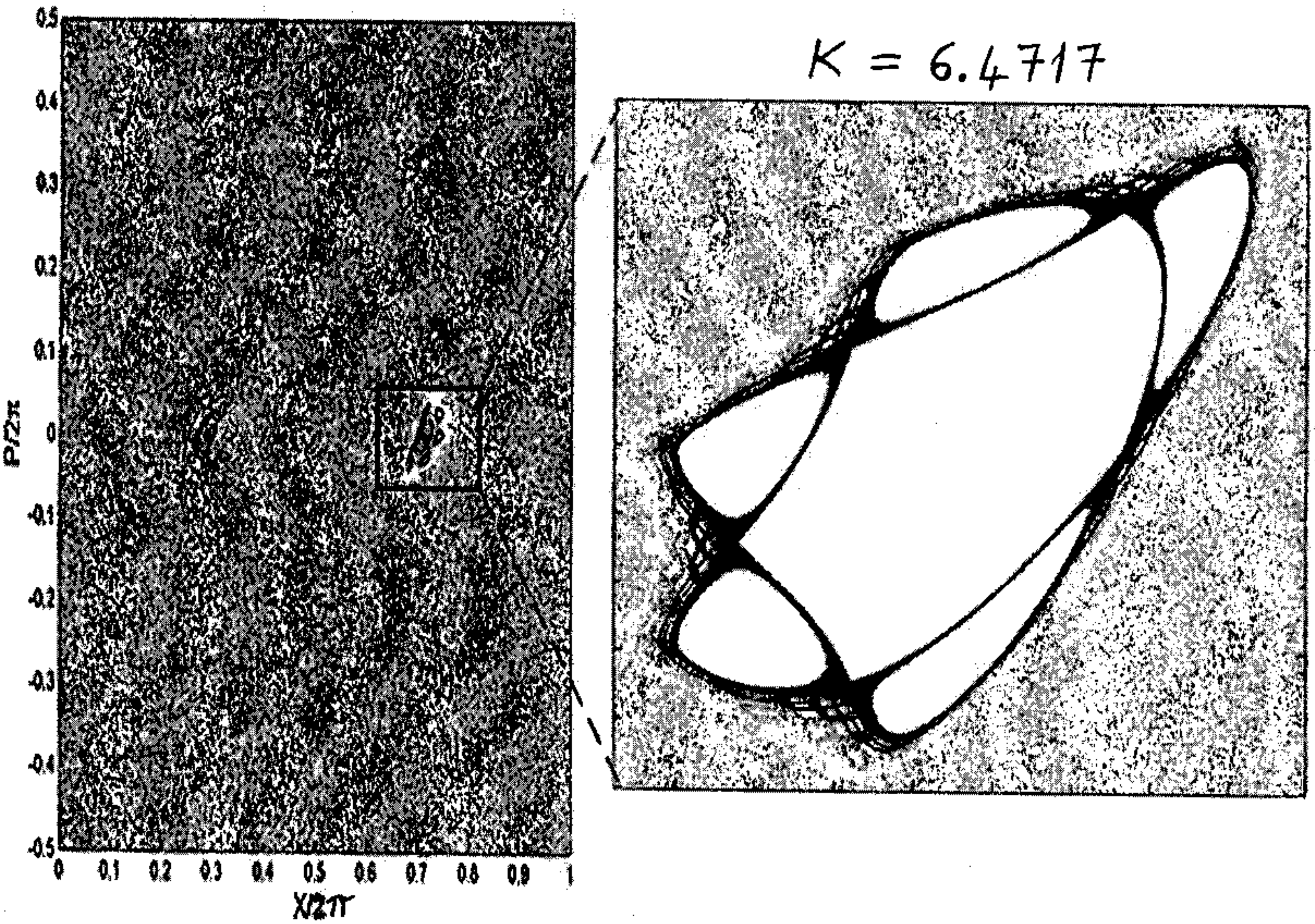
HAMILTONIAN WEAK – CHAOS SUPERDIFFUSION

Introduction

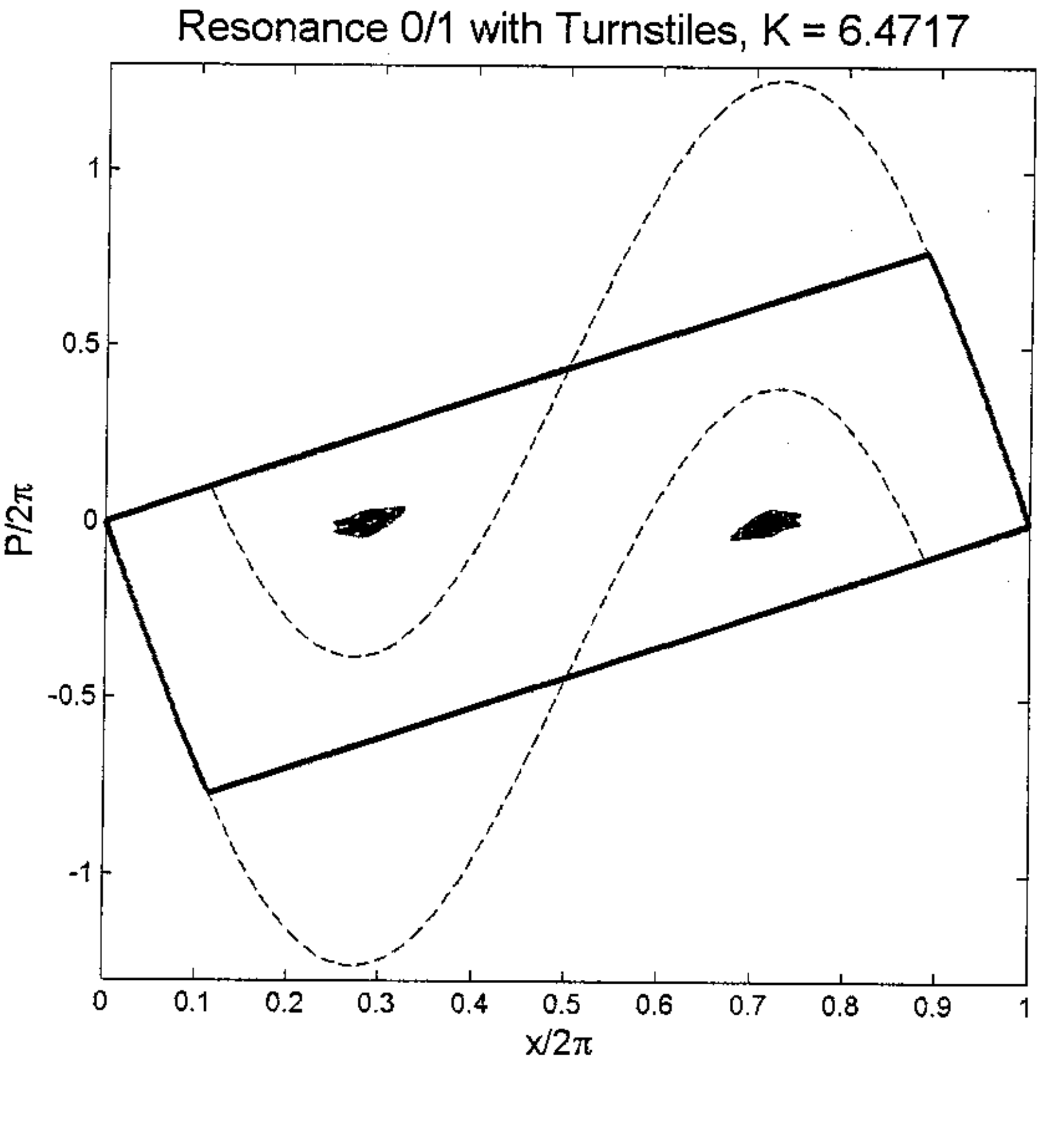
Standard map:

$$p_{n+1} = p_n + K\sin(x_n), \quad x_{n+1} = x_n + p_{n+1} \bmod 2\pi,$$

exhibits *superdiffusion,* $\langle p_n^2 \rangle \propto n^{\mu}$, $1 < \mu < 2$, in *strong*-chaos regime ($K > 2\pi$), due to *stickiness* to islands surrounding accelerator-mode fixed points, $p_1 = p_0 \pm 2\pi$, $x_1 = x_0$. Chaotic *"flights"*: $p_n \approx p_0 \pm 2\pi n$.

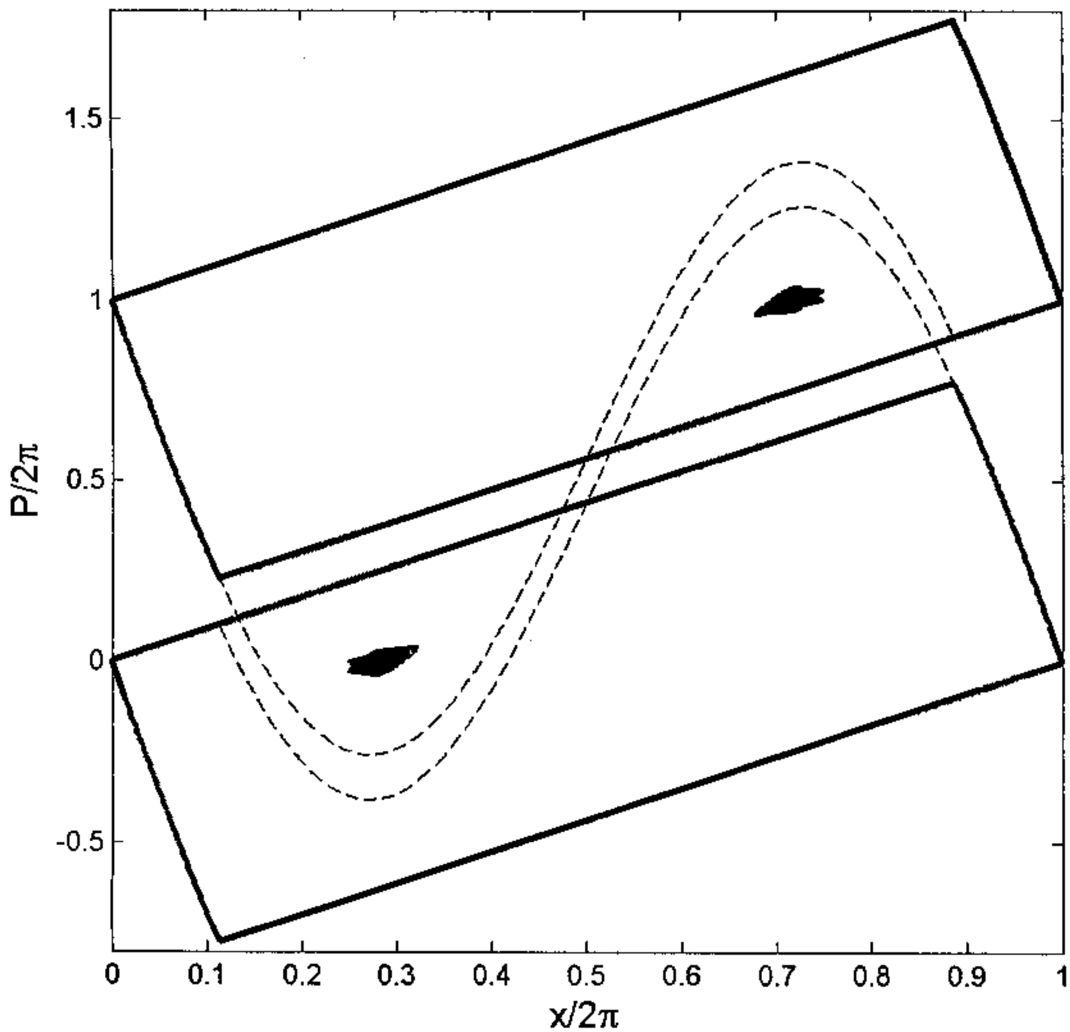


Accelerator-mode islands (AIs) are *"tangle"* islands, in the lobes of *resonance turnstiles*, and have *no* integrable limit.

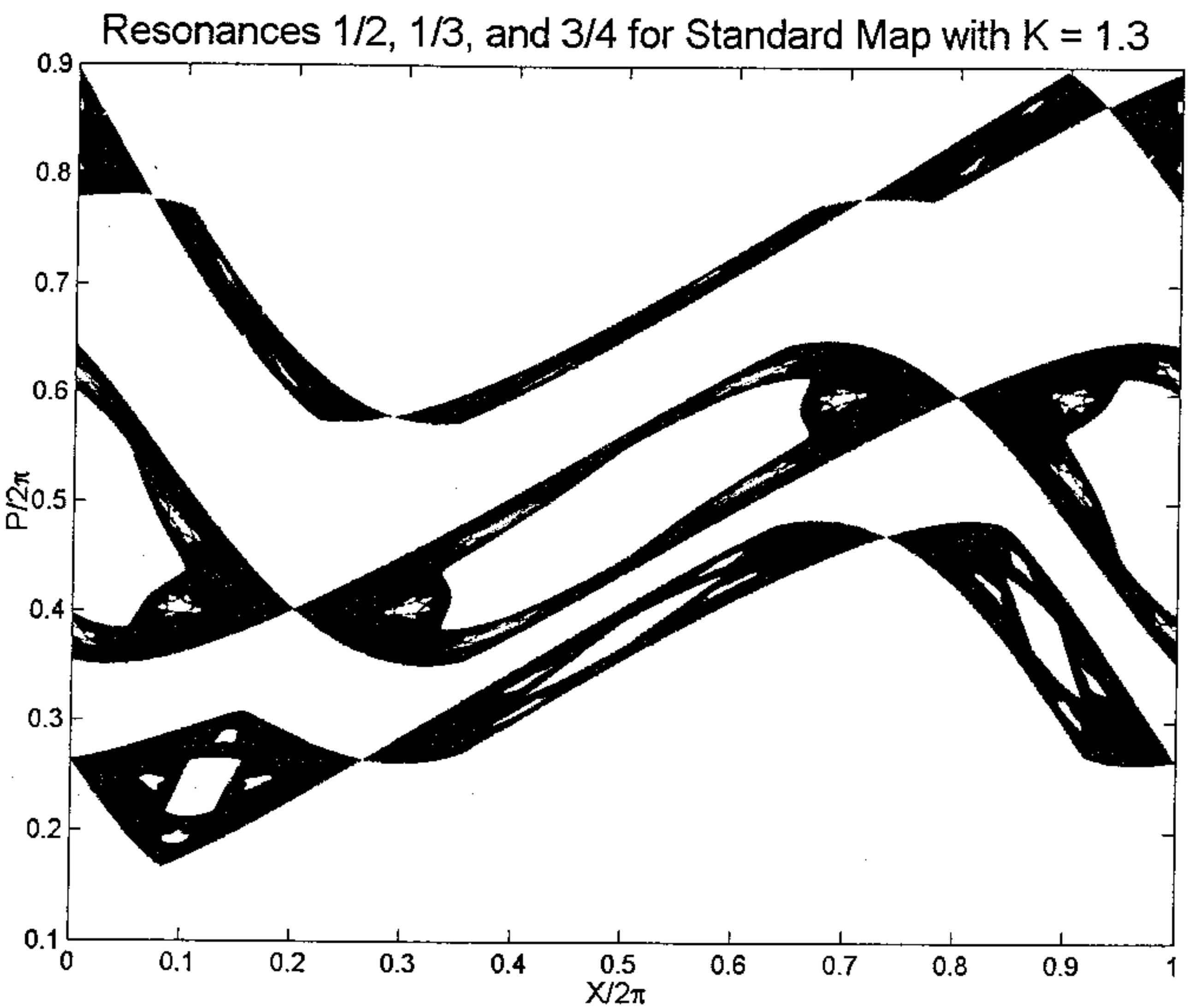


Actually, AIs must lie within the *turnstile overlap* of two neighboring first-order resonances (Dana & Barash, 2004).

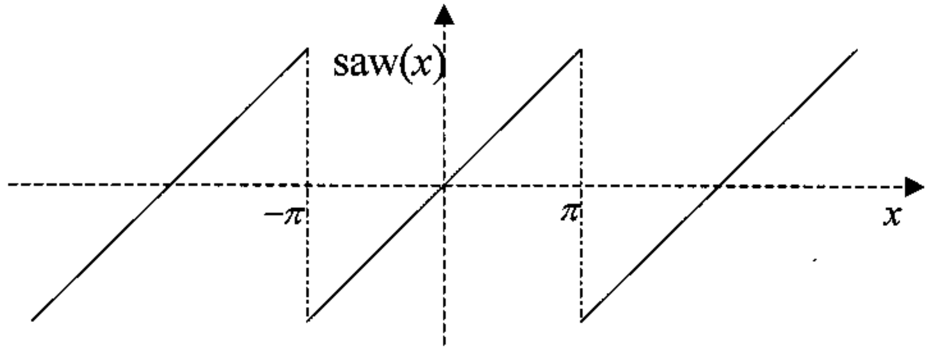
Resonances 0/1 and 1/1 with Turnstile Overlaps, K = 6.4717



MacKay, Meiss, & Percival (1987); Dana (1993).



Sawtooth Map: $p_{n+1} = p_n + \kappa\,\mathrm{saw}(x_n), \quad x_{n+1} = x_n + p_{n+1} \mod 2\pi$.



Hyperbolic case ($\kappa > 0$) [Rokhlin (1961), Chirikov (1969), Aubry (1978), Percival (1979)]: Totally chaotic but with nontrivial symbolic dynamics and featuring unstable *ordered orbits and resonances*, completely analogous to those in standard map.

- Markov model of transport on resonances and diffusion-coefficient scaling, $D \propto \kappa^{2.5}$, for small κ [Dana, Murray, & Percival (1989)].
- Organization and quasiregularity of chaos [Dana (1990, 1993)].
- Approximations of stadium billiard [Casati et al. (1996, 1999)].
- Quantized sawtooth map as model for quantum computation [Benenti et al. (2001, 2003), Bettelli and Shepelyansky (2003)].

<u>Elliptic case</u> $(-4 < \kappa < 0)$ [Ashwin (1997), Lowenstein, Vivaldi, and others (2000-2003)]: *Locally elliptic* map with rotation angle α,

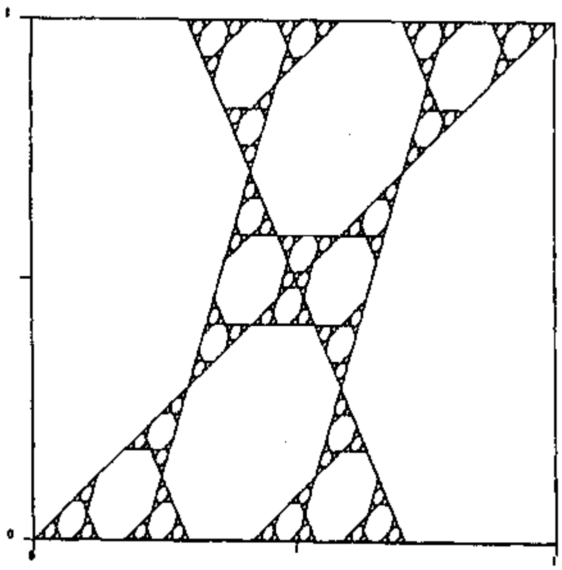
$$2\cos(\alpha) = \kappa + 2 \, .$$

A "*piecewise*" rotation, "taken modulo" the torus or the cylinder, featuring "*islands set*" and "*pseudochaotic region*" (PR):

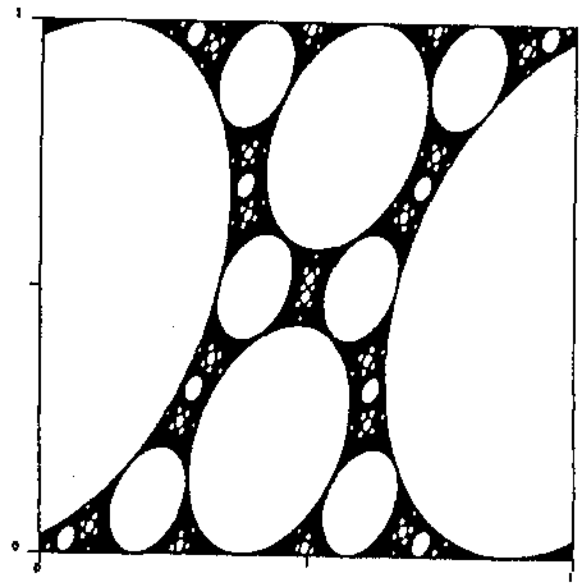
- *Integrable* (no PR) for $\kappa = -3,\ -2,\ -1$ $(\alpha = 2\pi/3,\ \pi/2,\ \pi/3)$.
- *Fractal* PR for noninteger κ and rational $\alpha/2\pi$.
- *Finite-measure* PR for irrational $\alpha/2\pi$.

Accelerator-mode fixed points $(x_0, p_0) = (\pm 2\pi/\kappa,\ 0)$ for $-4 < \kappa < -2$.

$\alpha = \pi/4$



Irrational $\alpha/2\pi$ $(\kappa = -0.5)$

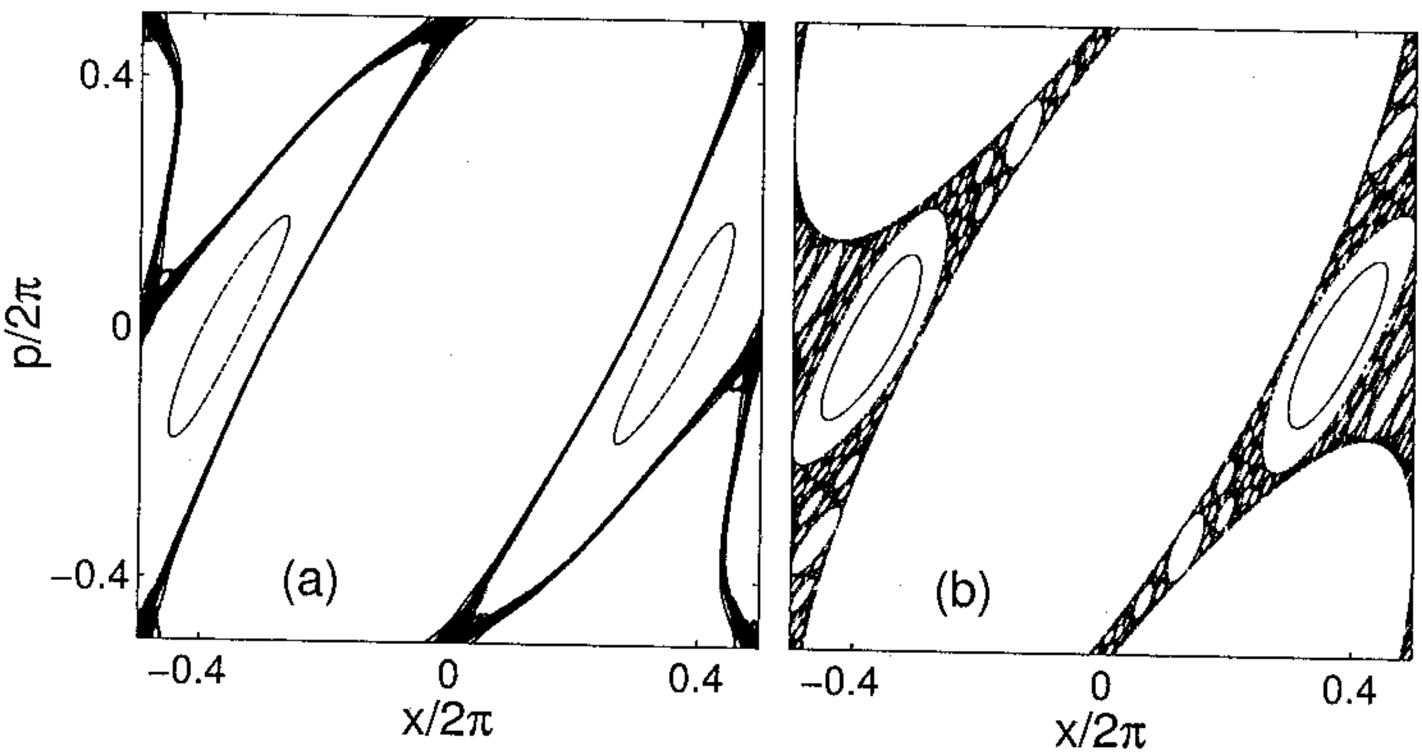


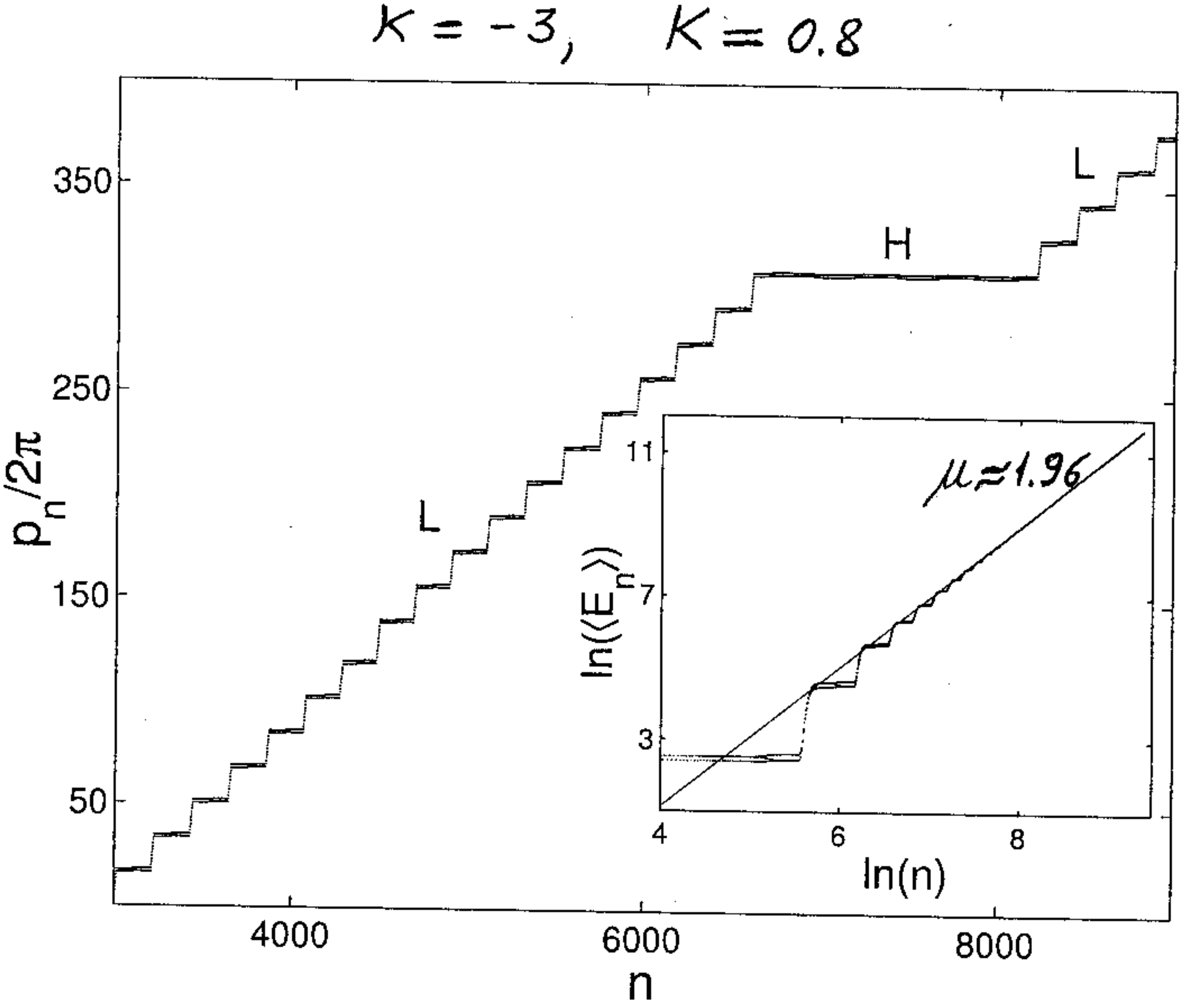
Perturbed Elliptic Sawtooth Map [Dana, PRE **69**, 016212 (2004)]:

$$\kappa\text{saw}(x) \longrightarrow \kappa\text{saw}(x) + K\sin(x)\,, \qquad -4 < \kappa < 0\,.$$

Accelerator-mode islands (AIs) and superdiffusion for $-4 < \kappa < -2$ and K *arbitrarily small* (*arbitrarily weak* chaos).

$\kappa = -3,\ K = 0.8$ $\qquad$ $\kappa = -2.7247498,\ K = 0.15$





K = -3, K = 0.8
350
250
150
50
$p_n/2\pi$
4000
6000
8000
n
L
H
L
11
7
3
$\ln(\langle E_n \rangle)$
4
6
8
ln(n)
$\mu \approx 1.96$

Precisely as the elliptic sawtooth map is a piecewise rotation, the perturbed elliptic sawtooth map (PESM) is a *piecewise "web map"*.

General web map on (u, v) *phase plane*:

Composition of *rotation* with *near-identity map*,

$$\Phi_{\text{web}}: \quad \begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \left[\begin{pmatrix} u_n \\ v_n \end{pmatrix} + K \begin{pmatrix} f(u_n, v_n) \\ g(u_n, v_n) \end{pmatrix} \right].$$

Special case: Kicked harmonic oscillator [Zaslavsky et al. (1986)].

Basic relation between PESM and Φ_{web}:

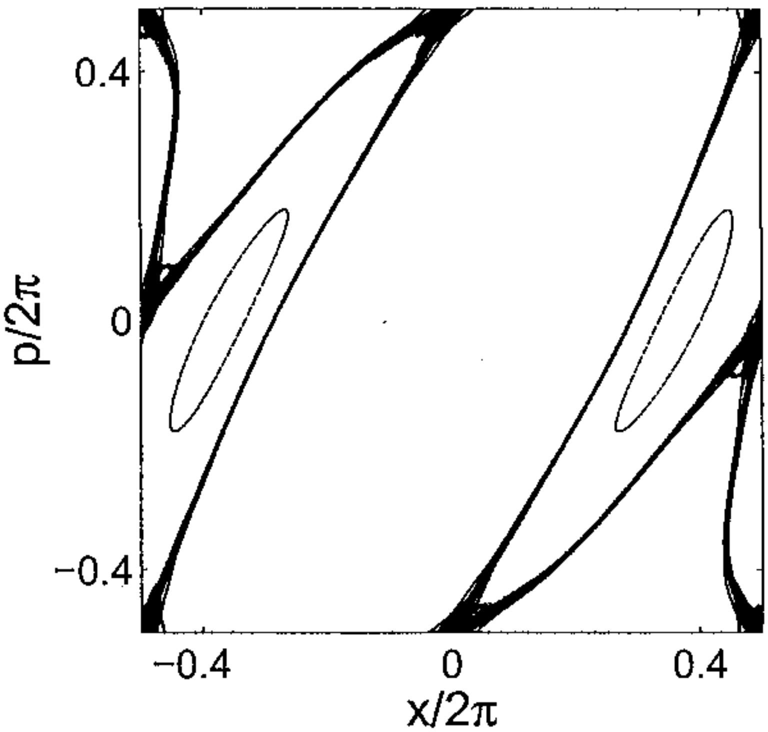
$$\text{PESM}(\kappa, K) = Q^{-1}\Phi_{\text{web}}Q \quad \text{modulo cylinder}, \quad \begin{pmatrix} x \\ p \end{pmatrix} = Q^{-1}\begin{pmatrix} u \\ v \end{pmatrix},$$

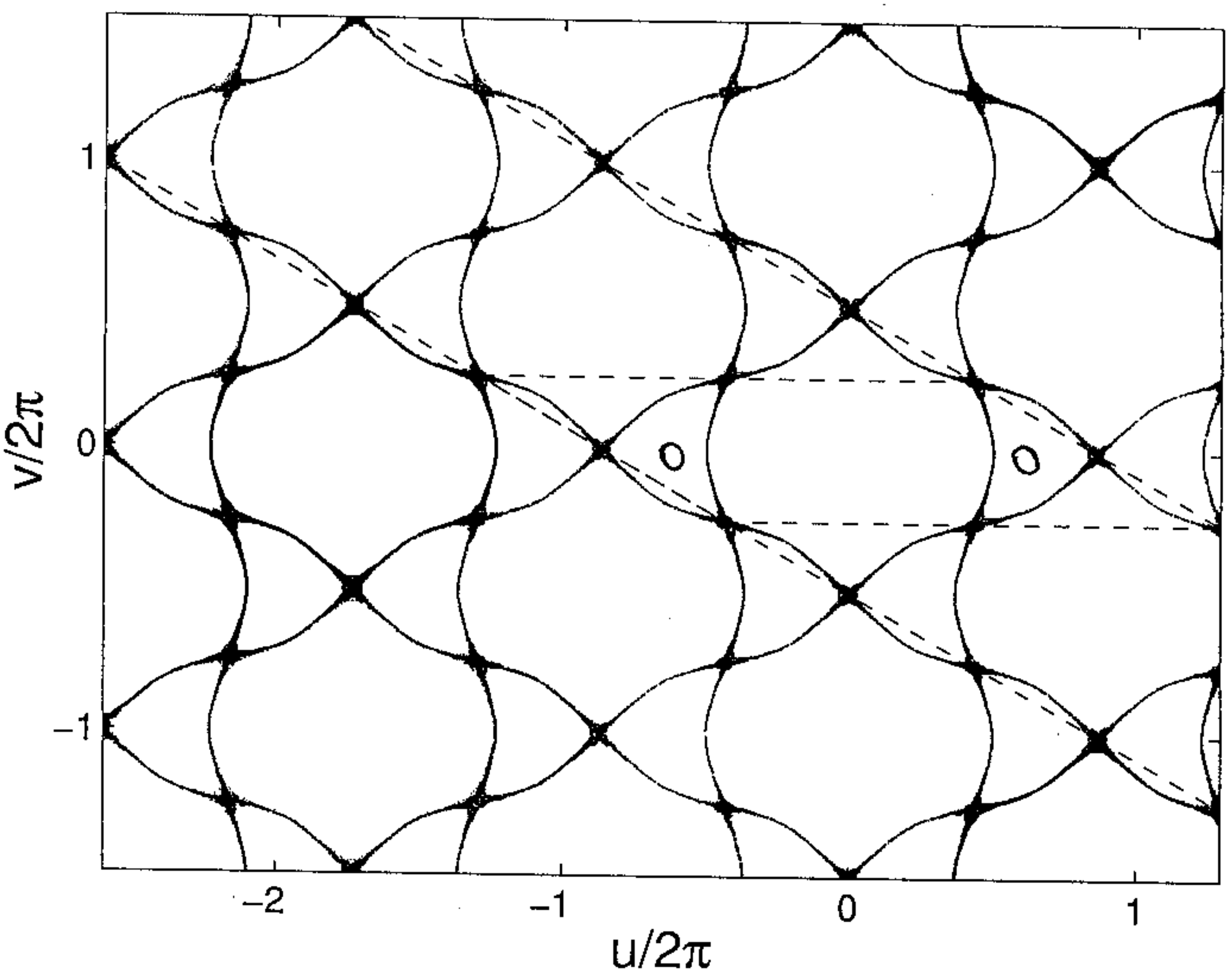
where again $2\cos(\alpha) = \kappa + 2$ and Q is a shear matrix:

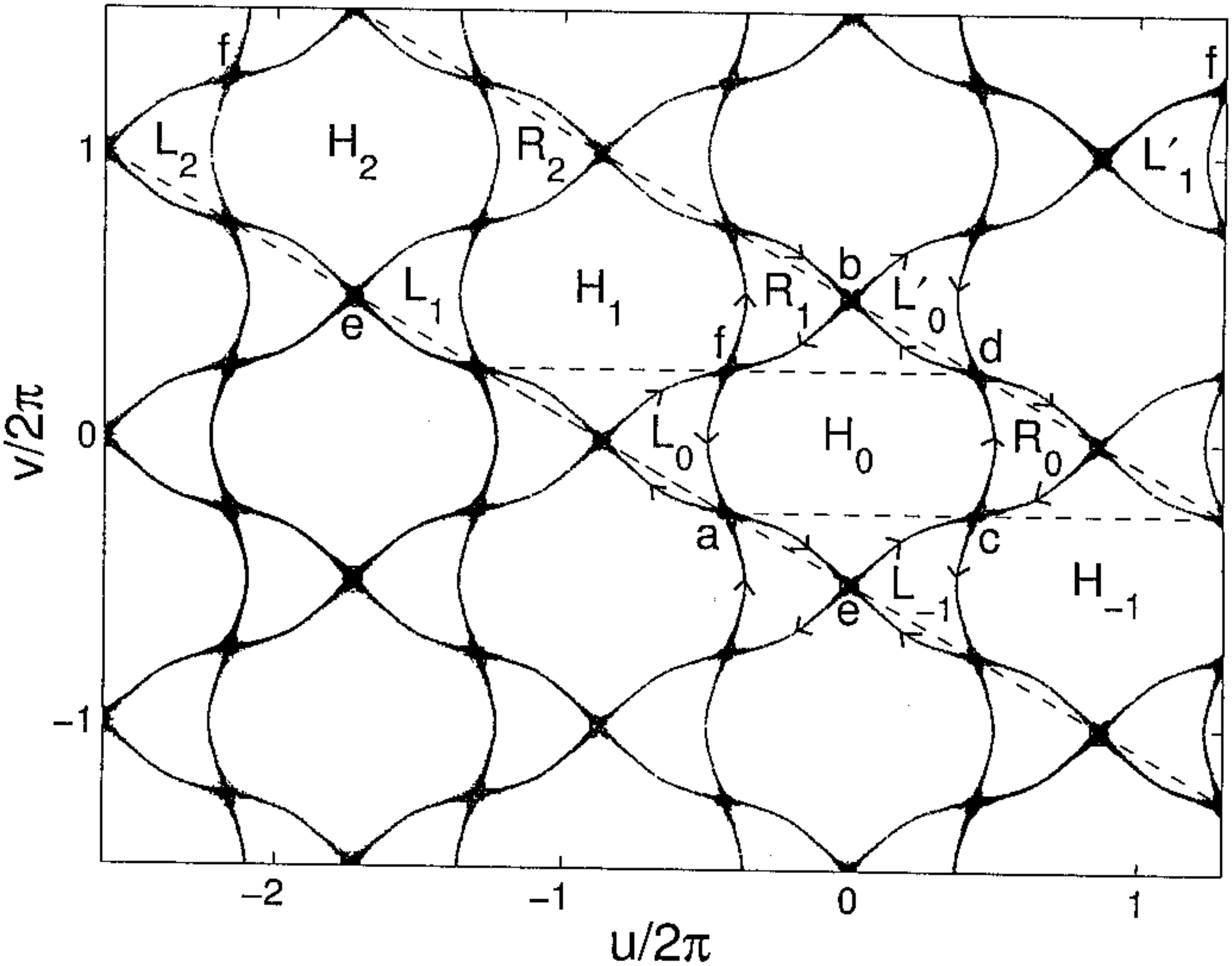
$$Q = \begin{pmatrix} \tan(\alpha/2) & -\tan(\alpha/2)/2 \\ 0 & 1/2 \end{pmatrix}.$$

PESM is essentially a web map "folded back" into the cylinder.

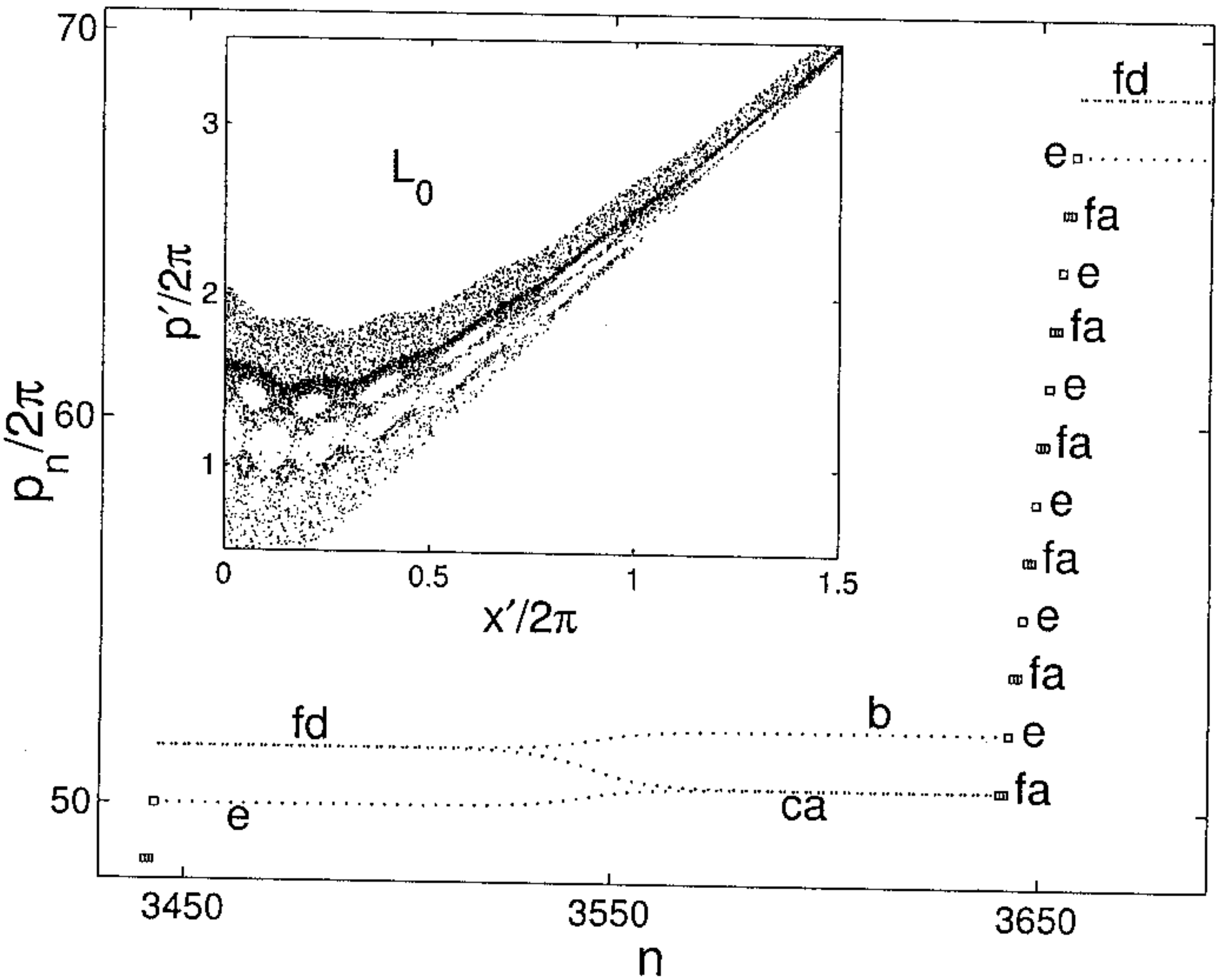
Example: $\kappa = -3, \quad K = 0.8.$



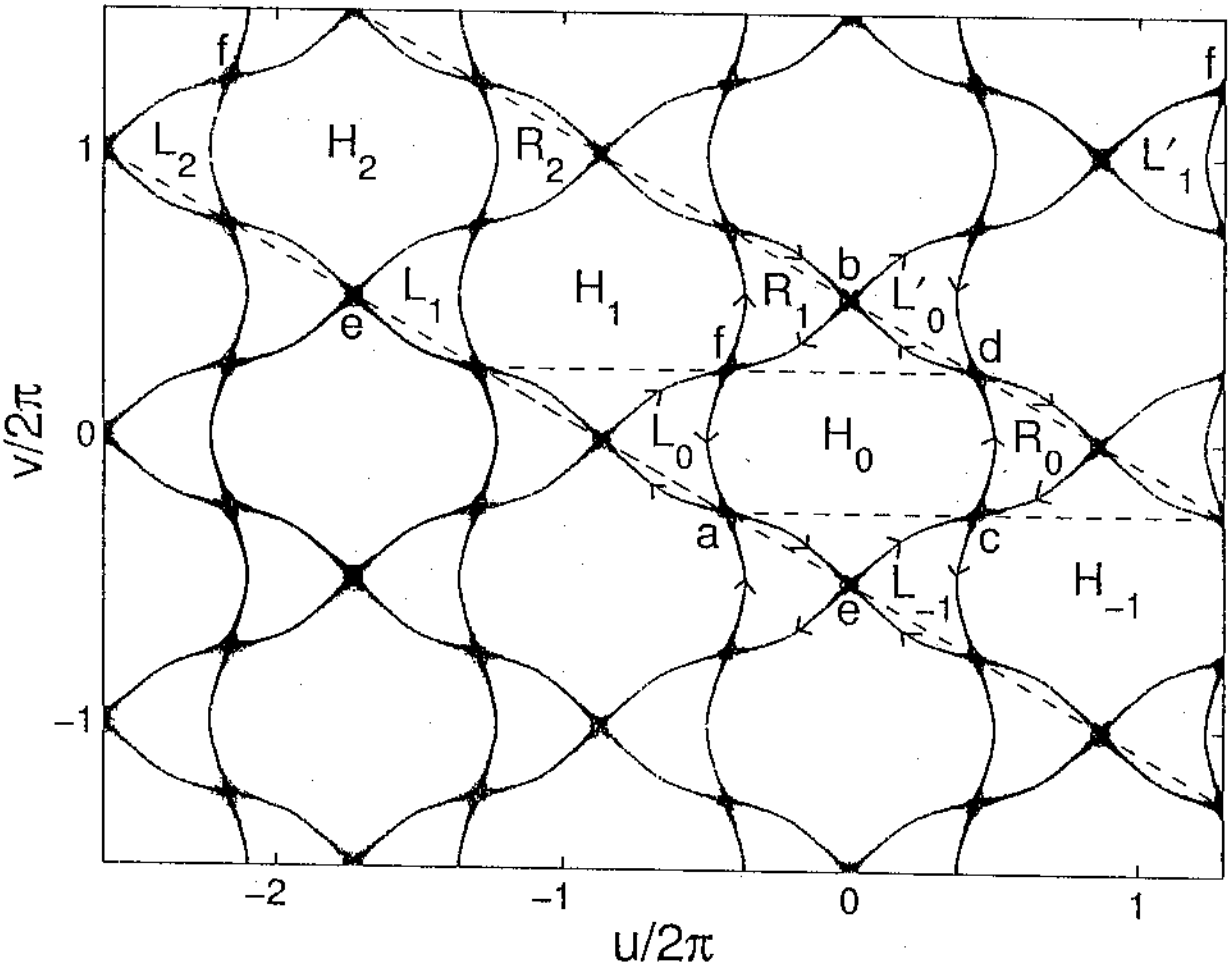




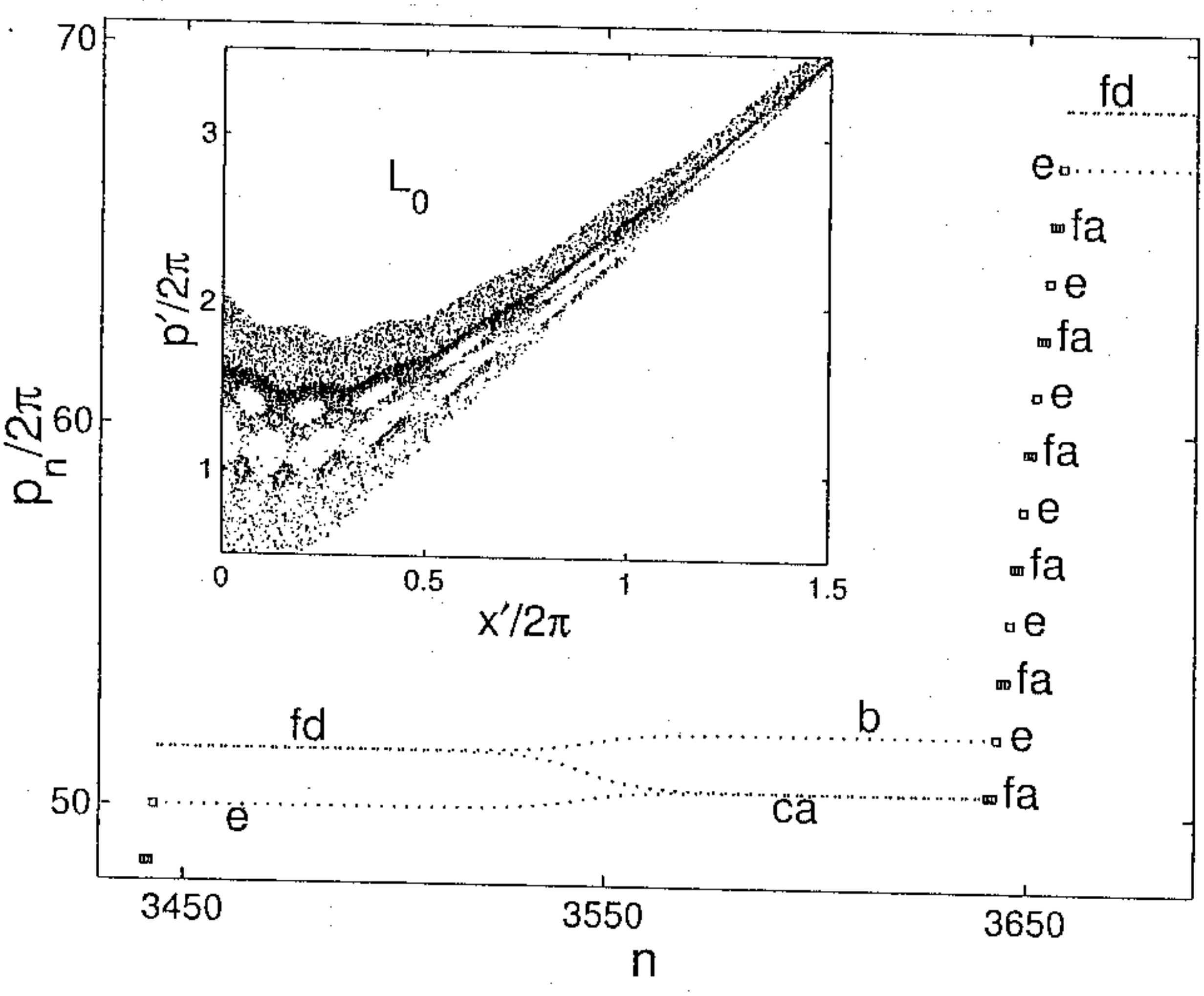
f
f
1
L
2
H
2
R
2
L′
1
b
L
1
H
1
R
1
L′
0
e
f
d
v/2π
0
L
0
H
0
R
0
a
c
L
−1
H
−1
e
−1
−2
−1
0
1
u/2π



70
60
50
$p_n/2\pi$
3450
3550
3650
n
fd
e
fa
e
fa
e
fa
e
fa
e
fa
e
fa
fd
b
e
ca
L_0
$p'/2\pi$
$x'/2\pi$
3
2
1
0
0.5
1
1.5



v/2π
u/2π
L2
H2
R2
L'1
L1
H1
R1
L'0
L0
H0
R0
L-1
H-1
a
b
c
d
e
f



p_n/2π
n
L0
p'/2π
x'/2π
fd
e
fa
b
ca

Conclusions

- Apparently first example of AIs (in PESMs) having *integrable* limit: essentially *normal* (web) islands "folded back" into the cylinder, thus fundamentally different from usual AIs ("tangle" islands).

- In weak-chaos regime, the boundary of an AI behaves mostly like that of a normal (web) island, and acceleration take places only when a chaotic orbit passes through tiny "*acceleration spots*" on the boundary.

- Consequence: Chaotic flights having a *steplike* structure, becoming increasingly pronounced in weak-chaos limit ($K \to 0$).

- Weak-chaos superdiffusion arises by "folding back the web-map normal diffusion into the cylinder".

- Work in progress: Quantization of PESMs.