Loschmidt Echo for a Chaotic Oscillator

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Technion

Def: Loschmidt Echo – fidelity of WF

WF at time = 0 $\Psi(0)=|\alpha\rangle$

WF at time = t $\Psi(t)=\widehat{U}(t)|\alpha\rangle$

Evolution operator $\widehat{U}(t)=\hat{\exp}[-\frac{i}{\hbar}\int_0^t H(\tau)d\tau]$

The exponent is T-ordered; $\hat{\exp}$

Rivers back at time t with a variant Hamiltonian

$$H_\delta(t)=H(t)+\delta H(t)$$

$$\widehat{U}_\delta^+(t)=\hat{\exp}[\frac{i}{\hbar}\int_0^t H_\delta(\tau)d\tau]$$

A mesure of reversibility is the fidelity of WF

$$M_\delta(t)=|\langle\alpha|\widehat{U}_\delta^+(t)\widehat{U}(t)|\alpha\rangle|^2$$

A. Peres (`84)

- **Fidelity of WF was introduced in the field of QC as an analog of the classical stability of trajectories for chaotic dynamics:**

- **Standard map:** $\begin{pmatrix} \bar{I} = I + K \sin\theta \\ \bar{\theta} = \theta + \bar{I} \end{pmatrix}$ **Classical analog of the QKR**

K is the chaos control parameter $\left| \frac{\partial \bar{\theta}}{\partial \theta} - 1 \right| = K > 1$

Locally: $(\theta_0, \theta_0 + \delta\theta)$ $\delta\theta(t) \approx \exp(\Lambda t)$ $\Lambda = \ln K > 0$

Λ **is the Lyapunov exponent**

Loschmidt echo: $M_\delta \approx e^{-\Lambda t} + e^{-\Gamma t}$ **R. Jalabert, H. Pastawsky, (2001)**

$M_\delta \approx e^{-\Lambda t}$ $t < \tau_\hbar$ $\tau_\hbar = \frac{1}{\Lambda} \ln \frac{I_0}{\hbar}$ **the Ehrenfest time**

$M_\delta \approx e^{-\Gamma t}$ $\Gamma = \Gamma(\delta)$ **Jacquod, Silvestrov, Beenakker, (2001)**

Chaotic nonlinear oscillator: $H = H_0 + V$

ϖ **linear frequency**

μ **nonlinearity**

$$H_0 = \hbar \varpi a^+ a + \hbar^2 \mu (a^+ a)^2, \quad [a, a^+] = 1$$

$$V = -\hbar\varepsilon(a^+ + a) \sum_{n=-\infty}^{\infty} \delta(t - nT) = -\hbar\varepsilon(a^+ + a) g(t)$$

ε **is an amplitude of the perturbation with the period** T

The classical limit: $\hbar = 0 \quad \hbar a^+ a \to I \qquad \sqrt{\hbar}\varepsilon \to \tilde{\varepsilon}$

The chaos control parameter: $K = 4\mu\tilde{\varepsilon}TI = 4\tilde{\mu}\tilde{\varepsilon}\varpi T >> 1$

The system is more complicated than the QKR, but it is very convenient for an analytical treatment in the semiclassical limit.

$$U(t) = \exp\left[-\frac{i}{\hbar}\int_0^t (\hat{H}_0 + V(\tau))\, d\tau\right]$$

Dimensionless Planck's constant: $\kappa = \hbar\mu T$

time: $t \Rightarrow t / T$

$$\hat{U}(T=1) = \exp\left[-i\varpi T a^{+} a - i\kappa (a^{+} a)^{2}\right] \exp\left[i\varepsilon (a^{+} + a)\right]$$

$$e^{-i\varpi T a^{+} a} \,|\,\alpha\rangle = |\,e^{-i\varpi T}\alpha\rangle \qquad e^{i\varepsilon(a^{+}+a)}\,|\,\alpha\rangle = e^{i\frac{\varepsilon}{2}(\alpha^{*}+\alpha)}\,|\,\alpha + i\varepsilon\rangle$$

$$\hat{\exp}\left[-i\kappa \int_{0}^{t} (a^{+} a)^{2} d\tau\right] = \int \prod_{\tau} \frac{d\lambda(\tau)}{\sqrt{4\pi i \kappa}} e^{i\int_{0}^{t} d\tau \lambda^{2}(\tau)/4\kappa} \; \hat{\exp}\left[-i \int_{0}^{t} d\tau \lambda(\tau) a^{+} a\right]$$

$$\Psi(t) = \int \prod_{\tau} \frac{d\lambda(\tau)}{\sqrt{4\pi i \kappa}} e^{i\int_{0}^{t} d\tau \lambda^{2}(\tau)/4\kappa} \exp\left[i\varepsilon \int d\tau F(\alpha_{\lambda}(\tau), \alpha_{\lambda}^{*}(\tau))\right] |\,\alpha_{\lambda}(\tau)\rangle$$

$$\alpha_{\lambda}(t) = e^{-i\Phi_{\lambda}(t)}\left[\alpha + i\varepsilon \int_{0}^{t} g(\tau) e^{i\Phi_{\lambda}(\tau)} d\tau\right] \qquad \Phi_{\lambda}(t) = \int_{0}^{t} d\tau \left[\varpi T + \lambda(\tau)\right]$$

At the moment t dynamics is reversed back to t=0 with a random time dependent process added to the linear frequency: $\varpi \to \varpi_\eta = \varpi + \eta(t)/T$

$$H(t) \to H(\eta(t),t) \qquad \Psi^*_\eta(t) = \left[U(\eta,t)\,|\,\alpha\rangle\right]^+$$

The Loschmidt echo averaged over the Gaussian process $P[\eta - \overline{\eta}, \sigma]$ **with nonzero the first** $\overline{\eta}$ **and the second** σ **moments is**

$$M_\sigma(t) = \int \prod_\tau d\eta(\tau) P[\eta(\tau) - \overline{\eta}, \sigma] M(\eta(\tau),t)$$

$$M(\eta(\tau),t) = \left|\langle \Psi^*_\eta(t)\,|\,\Psi(t)\rangle\right|^2$$

In the semiclassical limit when $\kappa << 1$, $\lambda = \lambda_{cl} + \kappa\lambda_q$

the functional integration over $\prod_\tau d\lambda(\tau)$ **is carried out exactly**

For $\sigma << 1$ **and K>>1,** $$M_\sigma(t) \propto \exp\left[-\frac{\overline{\eta}^2}{\Lambda^2} e^{\Lambda t}\right]$$

$\sigma >> 1$ $$M_\sigma(t) \propto \sqrt{\frac{\Lambda}{\sigma}}\; e^{-\Lambda t}$$

The semiclassical approximation is valid until a breaking time τ_{scl} : $t<\tau_{scl}$

The Jacobian of the Hamiltonian flow in the SC limit is

$$J \approx 1 - 4\kappa^{1/2} t$$

$$\tau_{scl} \approx \frac{1}{2}\sqrt{t_H} = \frac{1}{4\kappa^{1/2}}$$

- The Loschmidt echo in the Lyapunov regime $M \sim e^{-\Lambda t}$ is obtained on the semiclassical time scale

$$t >> \tau_{\hbar}$$