25 slides (not counting skips)

THE ONSET OF SYNCHRONISM IN SYSTEMS OF GLOBALLY COUPLED CHAOTIC AND PERIODIC OSCILLATORS

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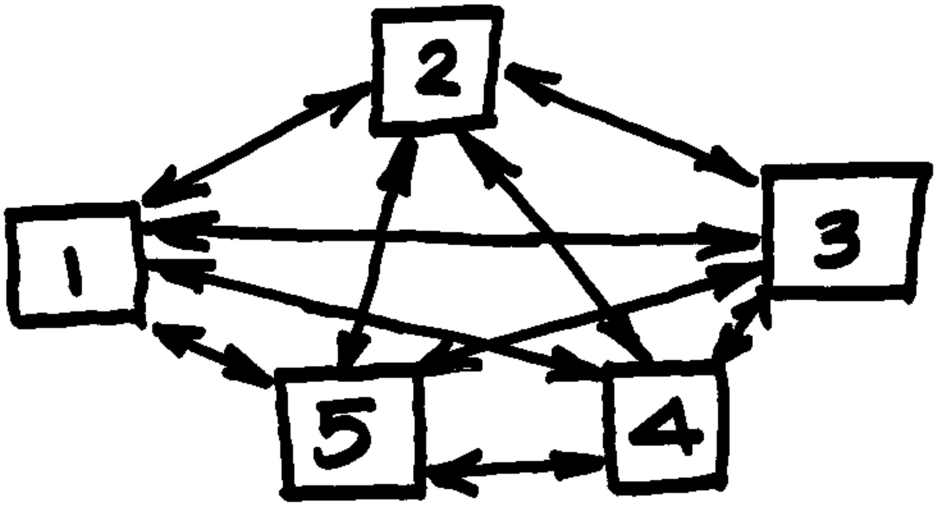
★ PHYSICA D 173, 29-51 (2002)
http://arXiv.org/abs/nlin.CD/0205018

GLOBALLY COUPLED SYSTEMS

N=5



□ = Dynamical system

- Dynamics of uncoupled system may be periodic (limit cycle attractor) or chaotic.

- We are interested in
  $N \gg 1$ (Numerical)
  $N \to \infty$ (Analytical)

PREVIOUS WORK

Limit cycles with spread of natural frequencies

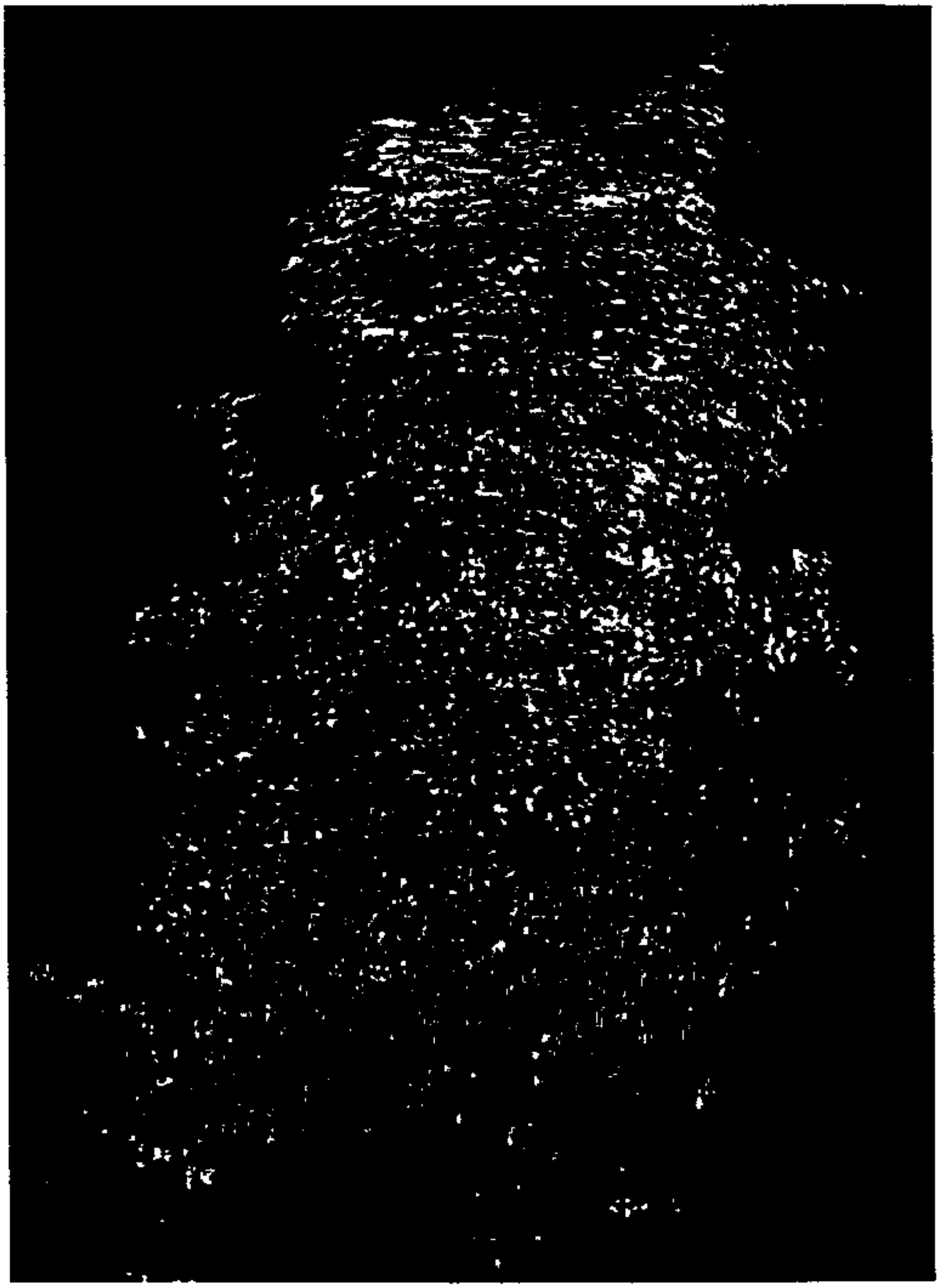
N. Weiner
Y. Kuramoto
A. Winfree
+ Many others

- Motivations: chirping crickets,
  flashing fire flies,
  Josephson junction arrays,
  Semiconductor laser arrays,
  cardiac pacemaker cells, etc

Globally coupled chaotic system

- Pikovsky, Rosenblum, Kurths, Eur.L. '9
- Sakaguchi, Phys. Rev. E '00.
- Topaj, Kye, Pikovsky, PRL '01

- Fireflies on a tree.



Review of the Onset of Synchrony in the Kuramoto Modle

(Ref. E. Ott 'Chaos in Dyn. ~~amical~~ Sys.' 2d Edition, Ch. 6, Sec. 5)

N coupled periodic oscillators whose states are described by phase angles θ_i ; $i = 1, 2, \ldots, N$.

$$d\theta_i/dt = \omega_i + \sum_j K_{ij}(\theta_j - \theta_i)$$

All-to-all sinusoidal coupling

$$K_{ij}(\theta) = (K/N)\sin\theta$$

- $$d\theta_i/dt = \omega_i + K\left\{\frac{1}{N}\sum_j \sin(\theta_i - \theta_j)\right\}$$

Complex Order Parameter

$$\left\{\frac{1}{N}\sum_j \sin(\theta_i - \theta_j)\right\} = \mathrm{Im}\left\{e^{-i\theta_i}\underbrace{\left[\frac{1}{N}\sum_j e^{i\theta_j}\right]}_{re^{i\psi}}\right\}$$

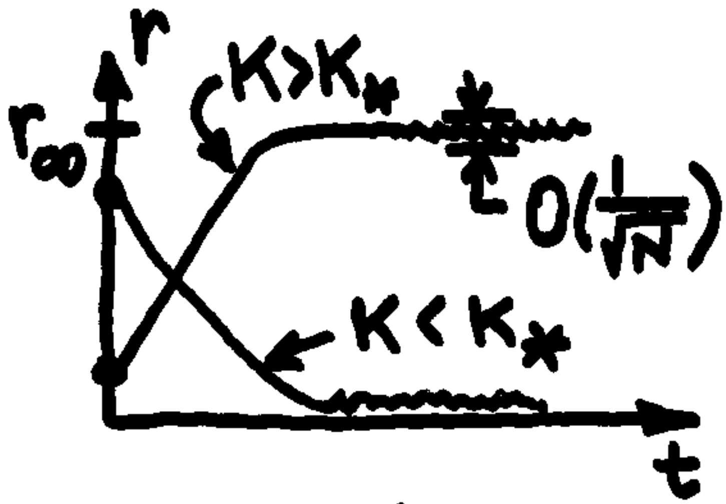
$$\boxed{\begin{aligned} d\theta_i/dt &= \omega_i + Kr\sin(\psi - \theta_i) \\ re^{i\psi} &= \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \end{aligned}}$$

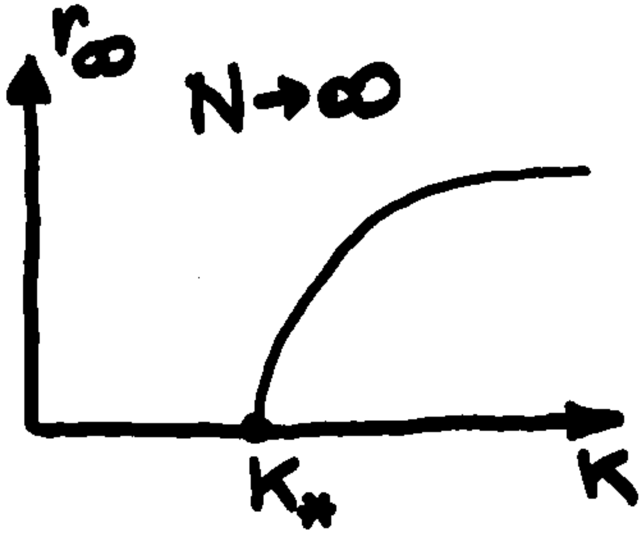
Typical Behavior

System specified by ω_i's and K.

Consider $N \gg 1$.

$G(\omega)\,d\omega$ = fraction of oscillation freqs. between ω and $\omega + d\omega$





$N \to \infty$

$\hat{F}(\theta, \omega, t)\,d\theta\,d\omega$ = fraction of oscillators whose phases and frequencies lie in the ranges θ to $\theta + d\theta$ and ω to $\omega + d\omega$.

$$\partial \hat{F}/\partial t + \partial/\partial\theta\left(\frac{d\theta}{dt}\hat{F}\right) + \partial/\partial\omega\left(\frac{d\omega}{dt}\hat{F}\right) = 0$$

$$\boxed{\begin{gathered}\partial \hat{F}/\partial t + \partial/\partial\theta\left[\left(\omega + Kr\sin(\psi - \theta)\right)\hat{F}\right] = 0 \\ re^{i\psi} = \int\int_0^{2\pi} \hat{F} e^{i\theta}\,d\theta\,d\omega\end{gathered}}$$

Linear Stability

Incoherent state:

$$\bar{F} = \frac{G(\omega)}{2\pi}, \quad 0 \leq \theta \leq 2\pi$$

This is a steady state solution.

Is it stable?

Linear perturbation: $\hat{F} = \bar{F} + f \;;\; \frac{|f|}{\bar{F}} \ll$

Laplace transform $\Rightarrow$

ODE in θ for f $\Rightarrow$

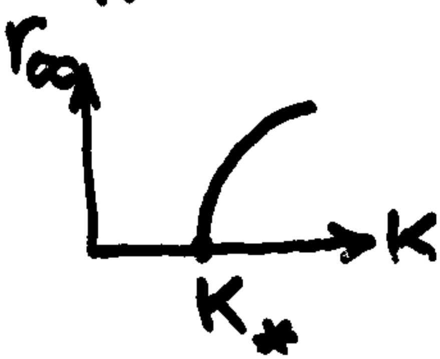
$D(s, K) = 0$ for given $G(\omega)$

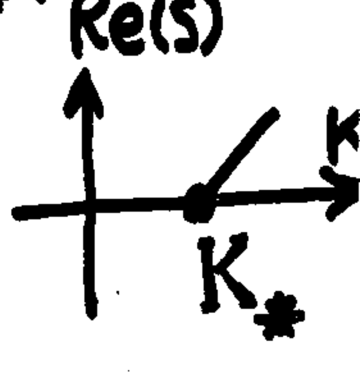
$\mathrm{Re}(s) > 0$ implies instability

Results: Critical coupling K_*.

Growth rates.

Oscillation freqs.





OUR WORK

- UNIFIES PREVIOUS WORK
- GIVES GROWTH RATES & K_*
- BASIC ISSUES (e.g., analytic continuation of dispersion function)
- NUMERICAL METHODS FOR COMPUTING DISPERSION FUNC
- EXAMPLES:

  KURAMOTO MODEL

  [redacted]

  LORENZ SYSTEMS

  r in $[r_-, r_+]$

A POTENTIALLY SIGNIFICANT RESULT

Even when the coupled units are chaotic systems that are individually not in any way oscillatory (e.g., 2x mod 1 maps or logistic maps), the global average behavior can have a transition from incoherence to oscillatory behavior (i.e., a supercritical Hopf bifurcation).

GLOBALLY COUPLED LORENZ SYSTEMS

$$dx_i/dt = \sigma(y_i - x_i) + k \langle\langle x \rangle\rangle$$

$$dy_i/dt = r_i x_i - y_i - x_i z_i$$

$$\langle\langle x \rangle\rangle = N^{-1} \sum_{i=1}^{N} x_i(t)$$

$$dz_i/dt = -b z_i + x_i y_i$$

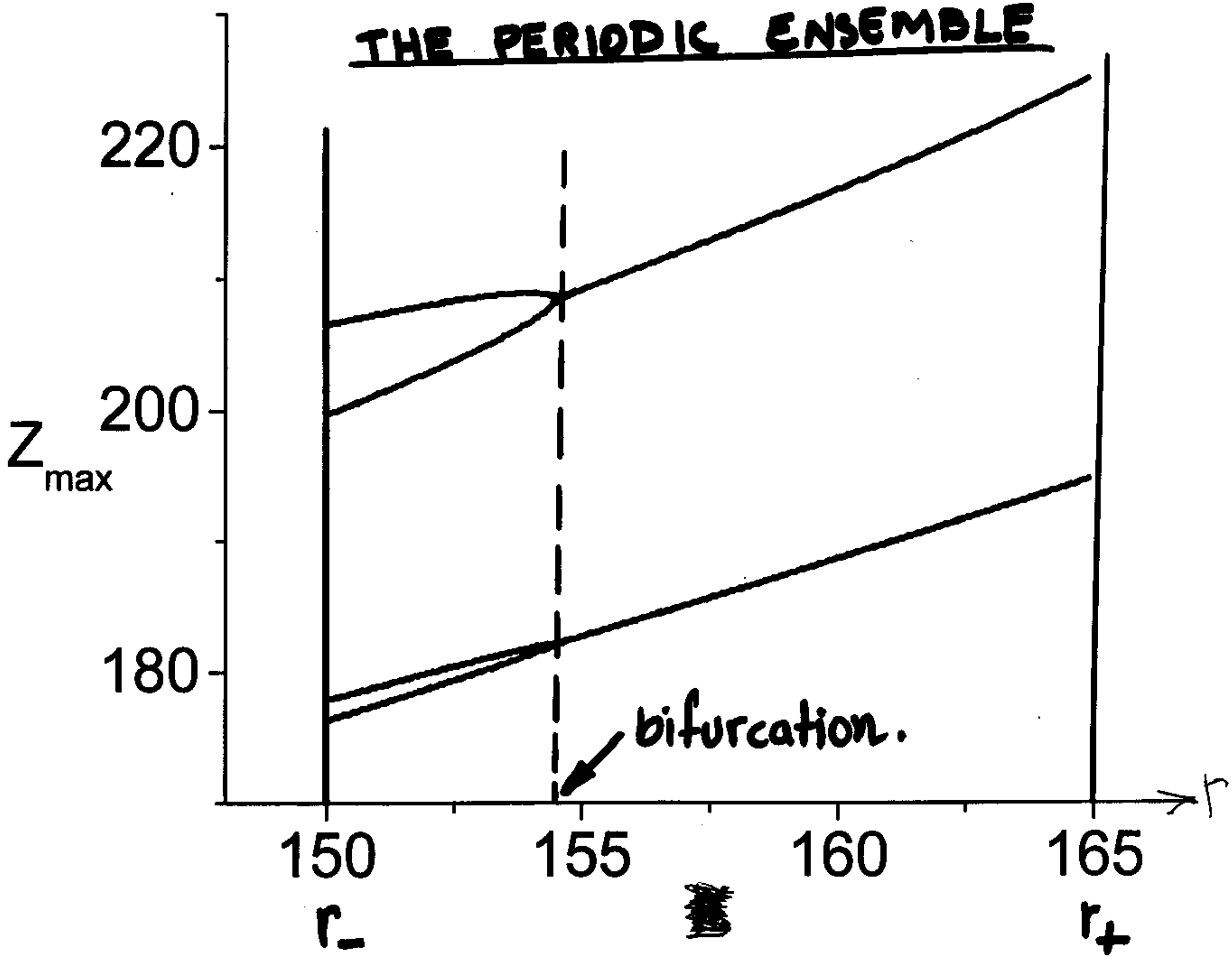
$$\sigma = 10 \; ; \; b = 8/3$$

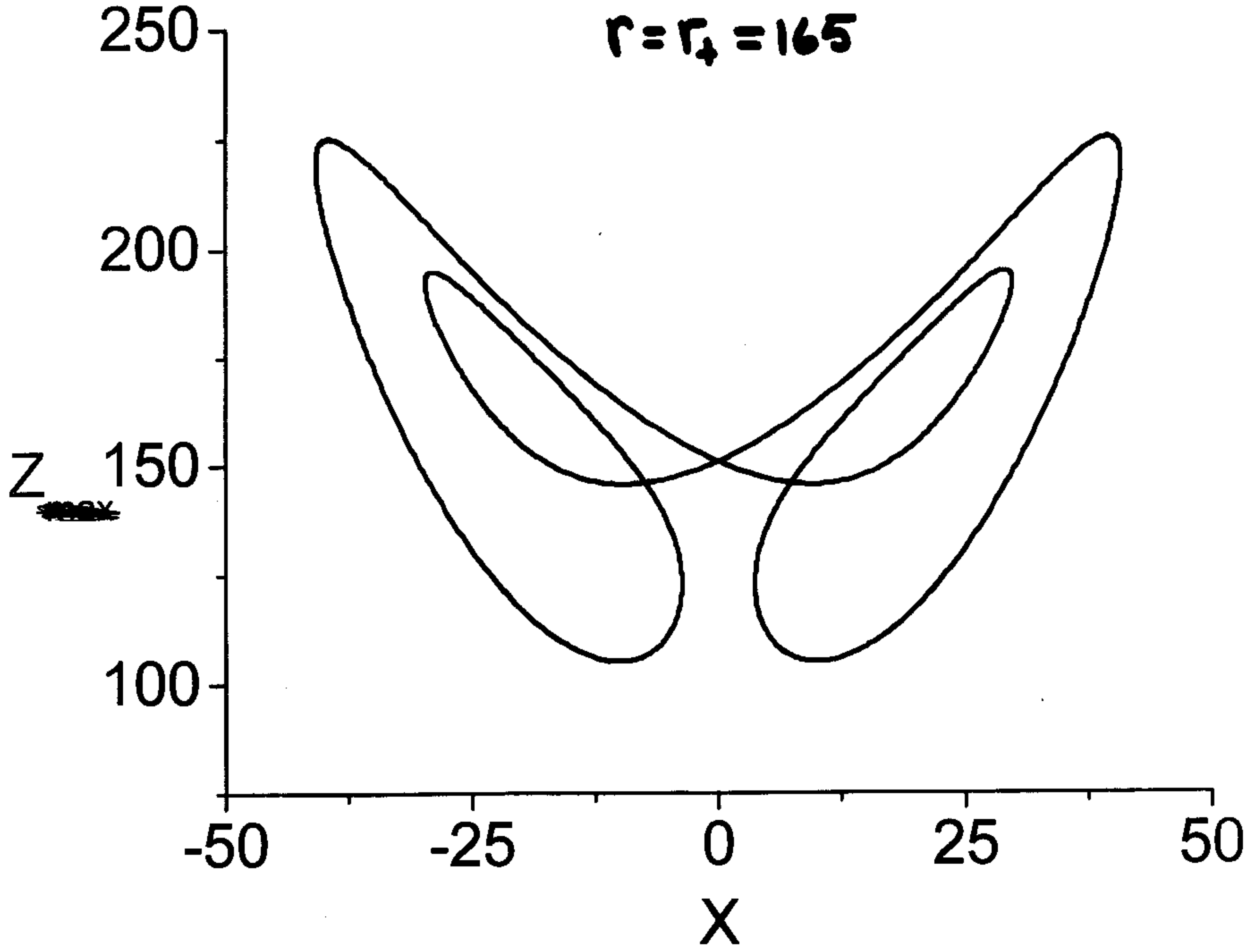
r_i uniformly distributed in $[r_-, r_+$

3 ENSEMBLES

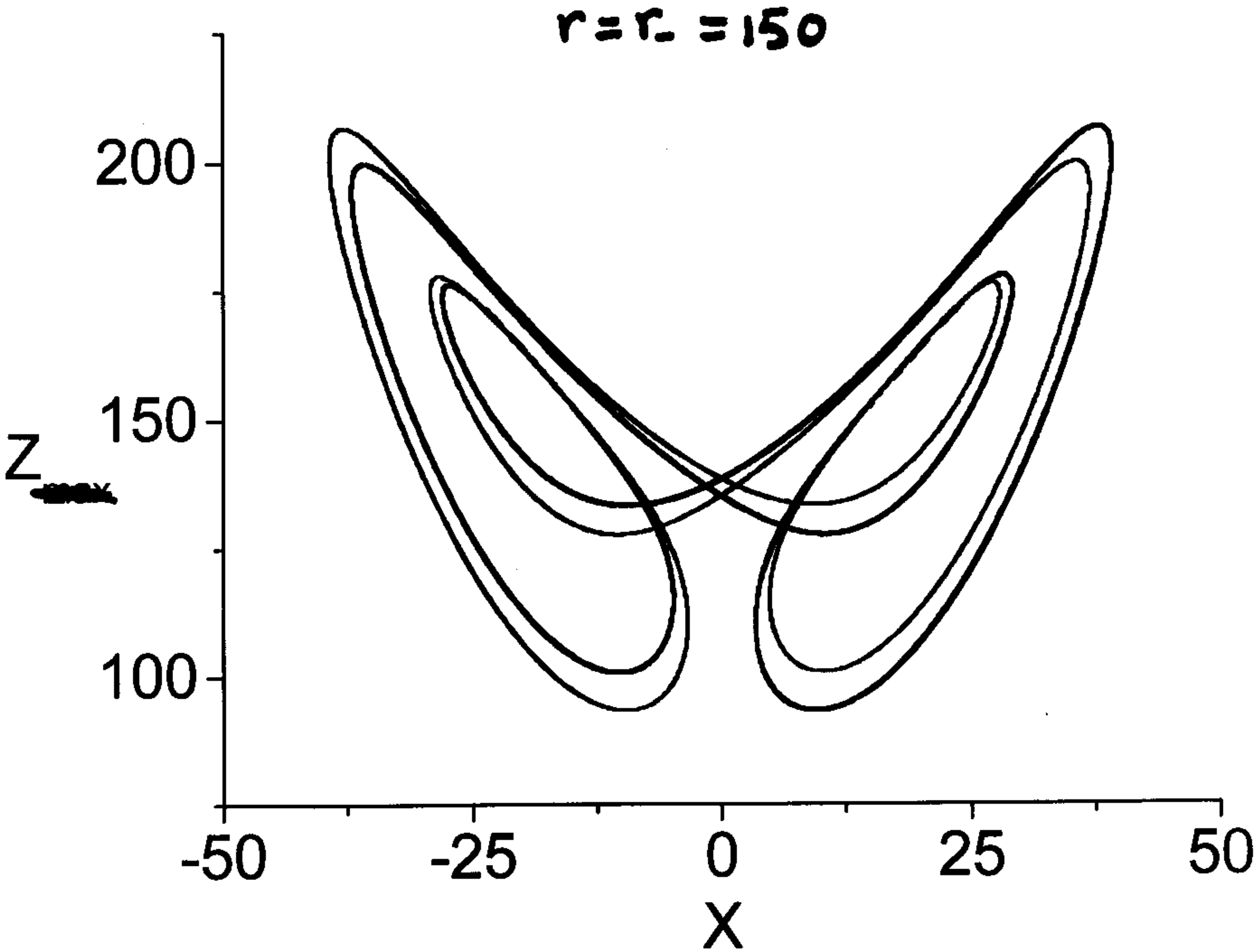
- Periodic ensemble: r in $[150, 16$
- Chaotic ensemble: r in $[28, 52]$
- Mixed ensemble: r in $[167, 202]$

BIFURCATION DIAGRAM FOR THE PERIODIC ENSEMBLE





r = r+ = 165
250
200
150
100
Z
-50
-25
0
25
50
X



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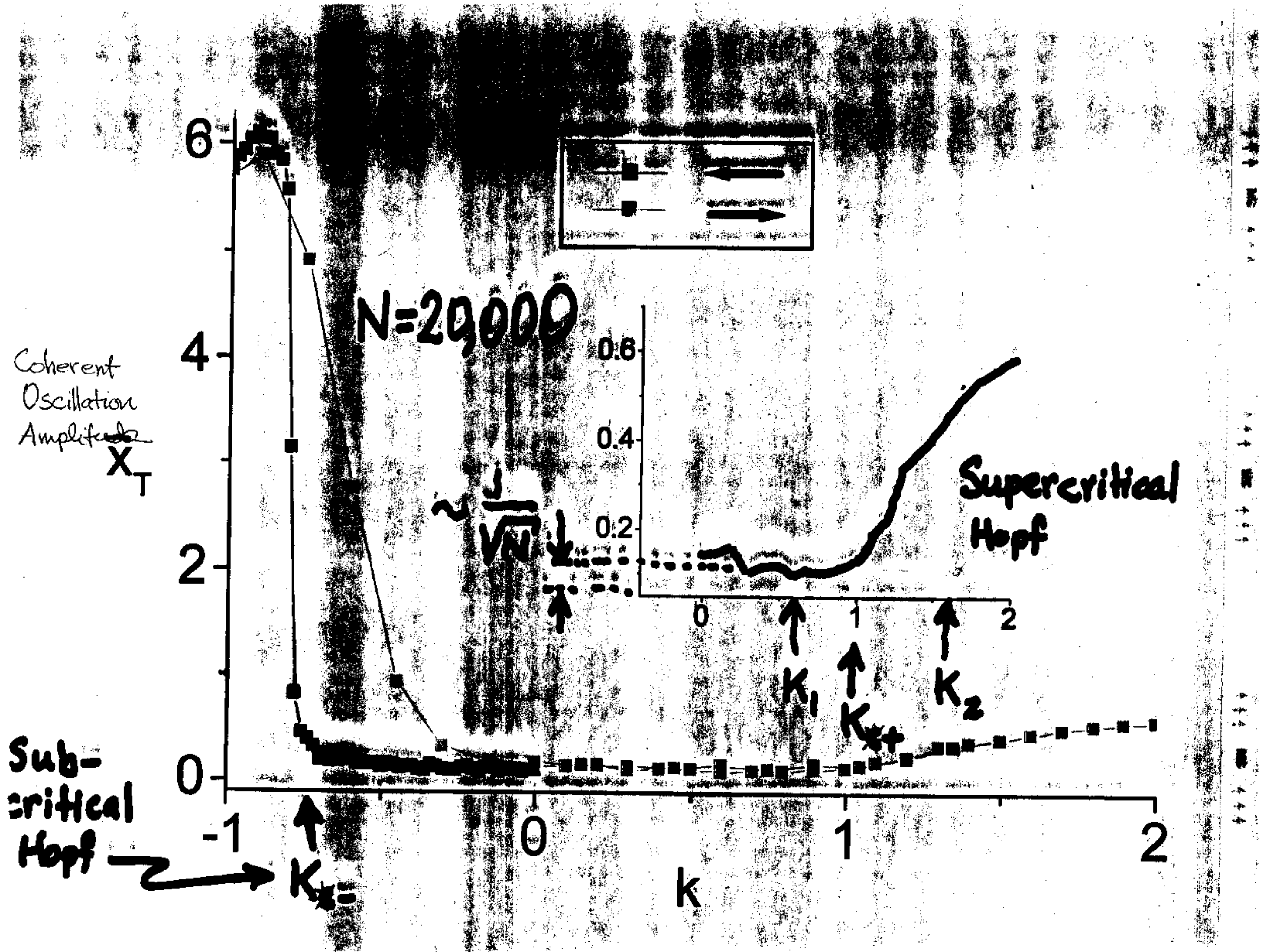
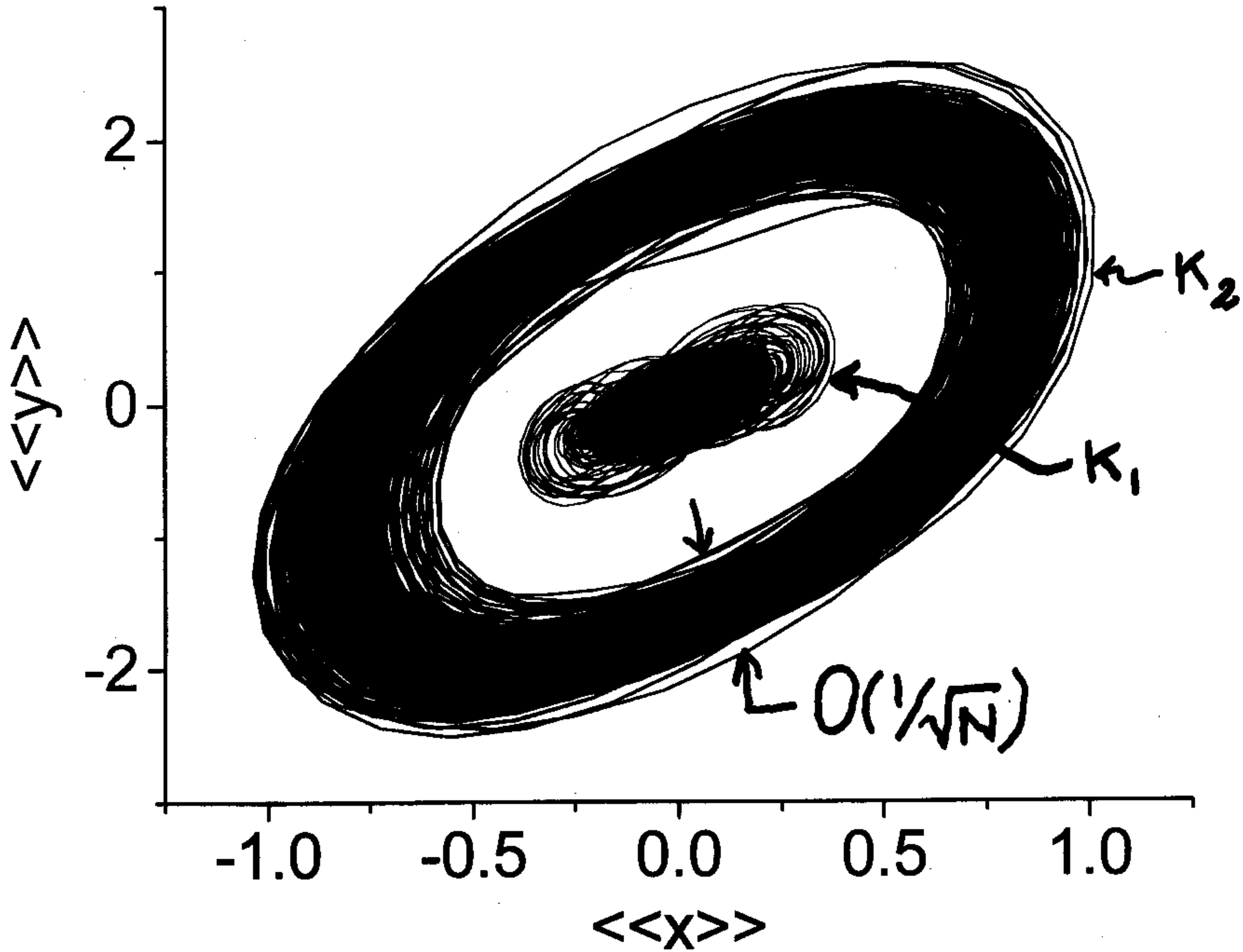


Coherent Oscillation Amplitude
X_T
6
4
2
0
-1
0
1
2
k
N=20,000
0.6
0.4
0.2
0
1
2
$\sim \frac{1}{\sqrt{N}}$
Supercritical Hopf
K_1
K_{x+}
K_2
Sub-critical Hopf
K_{x-}



2
0
-2
-1.0
-0.5
0.0
0.5
1.0
<<x>>
<<y>>
K2
K1
O(1/√N)

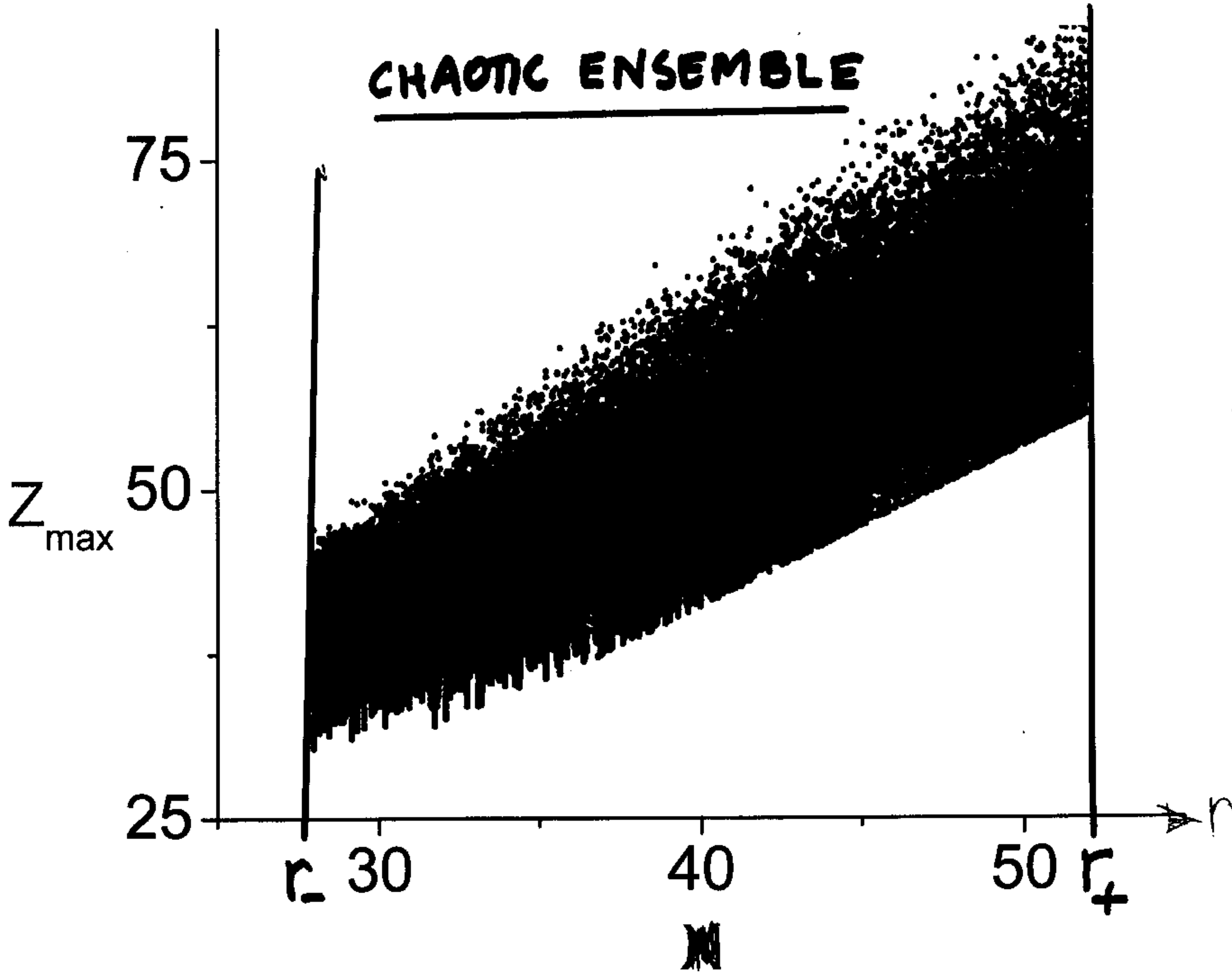
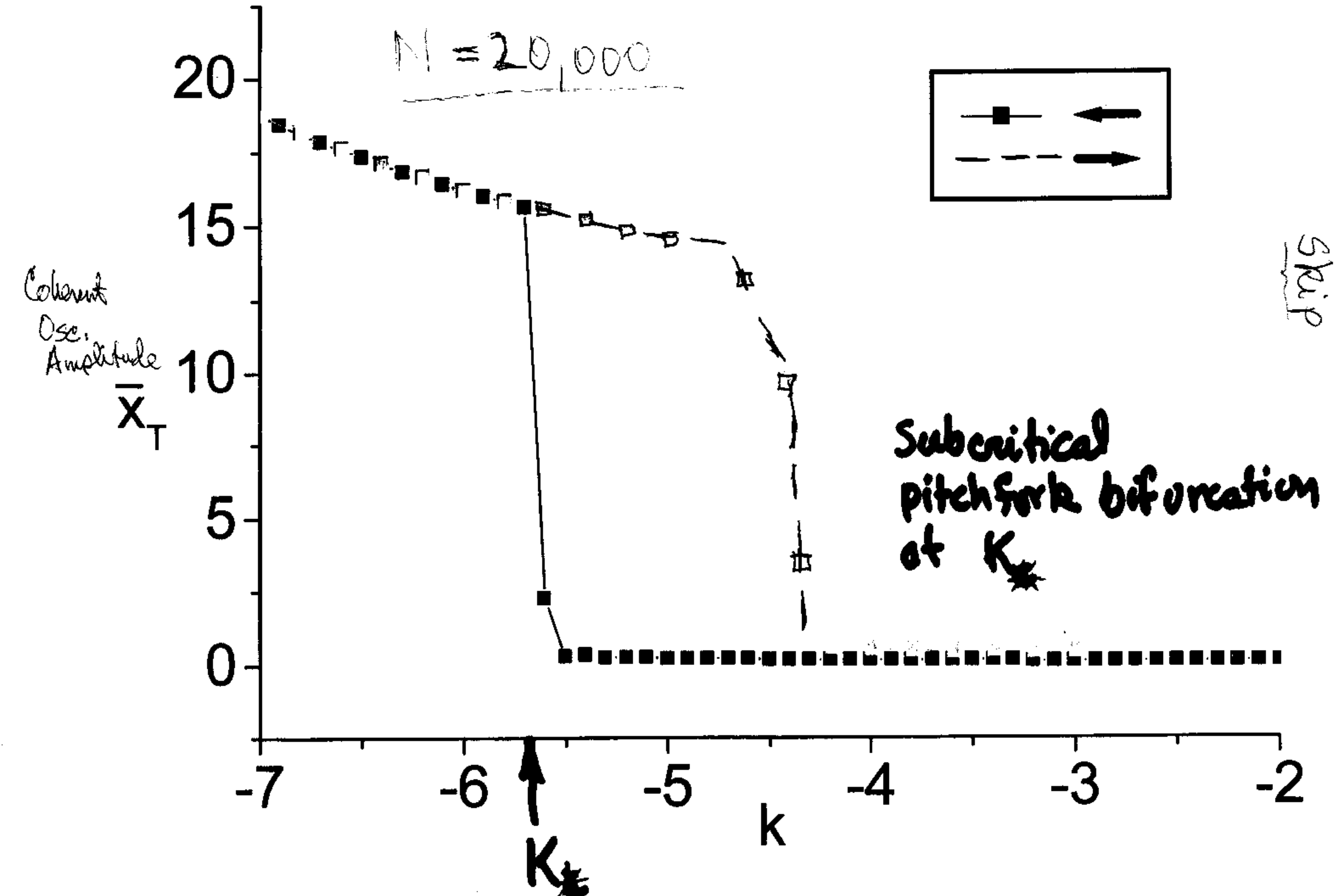
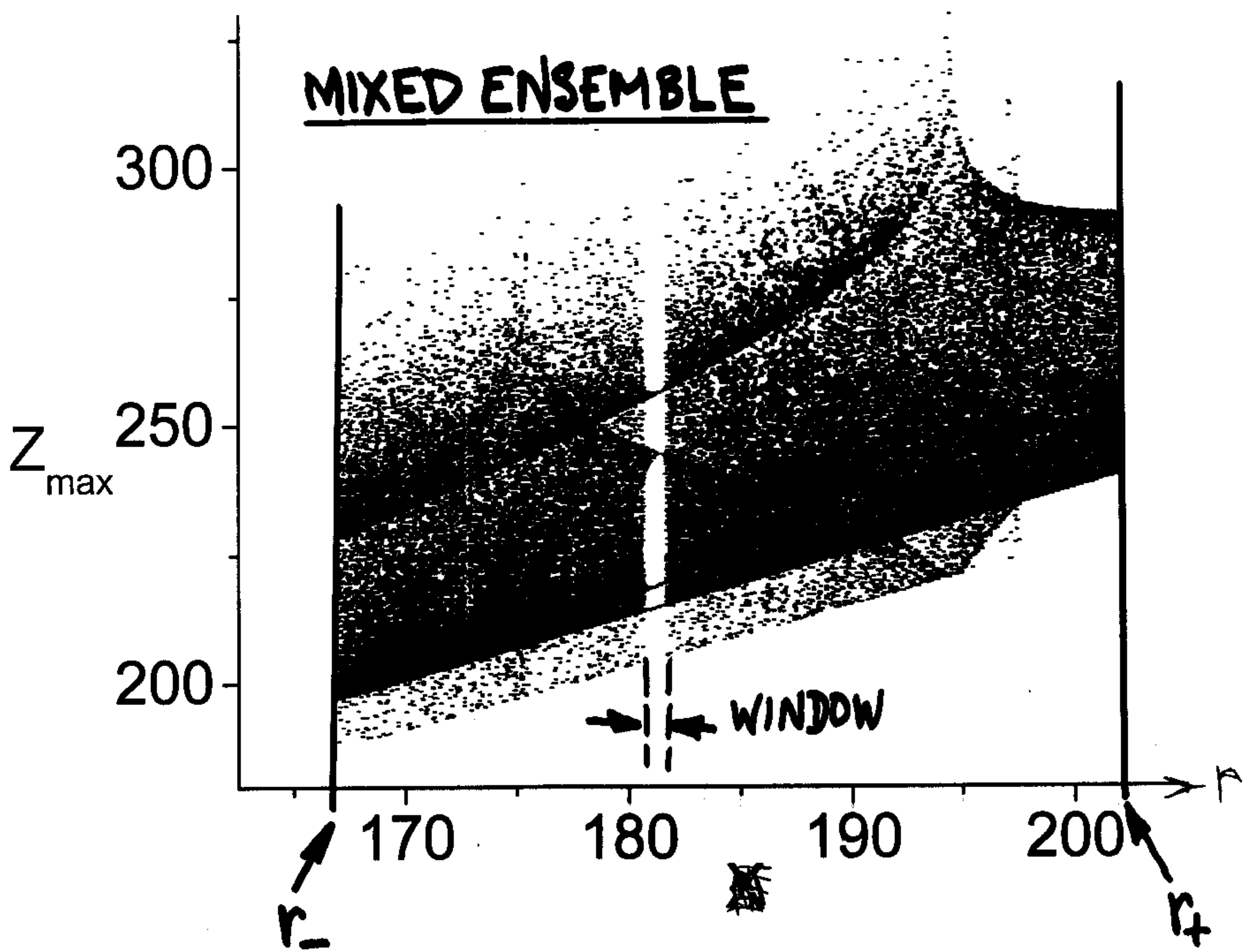


Fig. 6

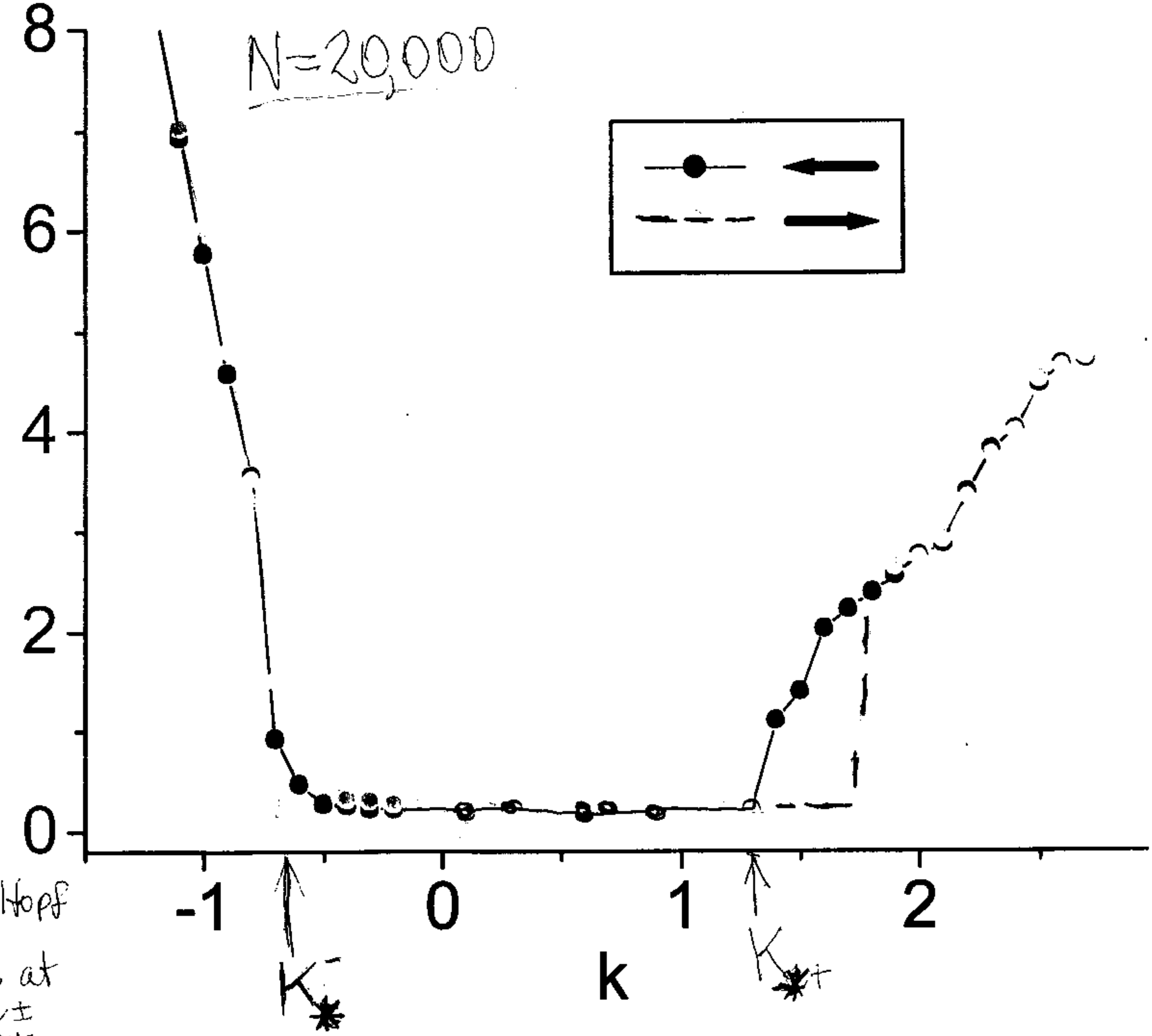

N = 20,000
20
15
10
5
0
Coherent Osc. Amplitude
$\bar{X}_T$
-7
-6
-5
-4
-3
-2
k
K_*
Subcritical pitchfork bifurcation at K_*
Skip



MIXED ENSEMBLE
300
250
200
Z
max
WINDOW
170
180
190
200
r
r_
r+

Coherent oscillation amplitude

$\bar{x}_T$



Supercritical Hopf bifurcations at $K = K_*^{\pm}$

FORMULATION

- $d\underline{x}_i(t)/dt = \underline{G}(\underline{x}_i(t), \underline{\Omega}_i) + \underline{\underline{K}} \cdot (\langle\langle \underline{x} \rangle\rangle_* - \langle\langle \underline{x} \rangle\rangle)$
- $i = 1, 2, \ldots, N$ lables ensemble members

  $\underline{\Omega}_i$ = parameter vector with smooth density distribution $\rho(\underline{\Omega})$

- $\underline{x}_i(t) = \left[x_i^{(1)}(t), x_i^{(2)}(t), \ldots, x_i^{(q)}(t) \right]^T$
- $\underline{\underline{K}} = q \times q$ coupling matrix
- $\langle\langle \underline{x}(t) \rangle\rangle = \lim_{N \to \infty} N^{-1} \sum_i \underline{x}_i(t)$
- $\langle\langle \underline{x} \rangle\rangle_* = \int \underline{x}\, \rho(\underline{\Omega})\, d\mu_{\underline{\Omega}}\, d\underline{\Omega}$
- $\mu_{\underline{\Omega}}$ = natural invariant measure.

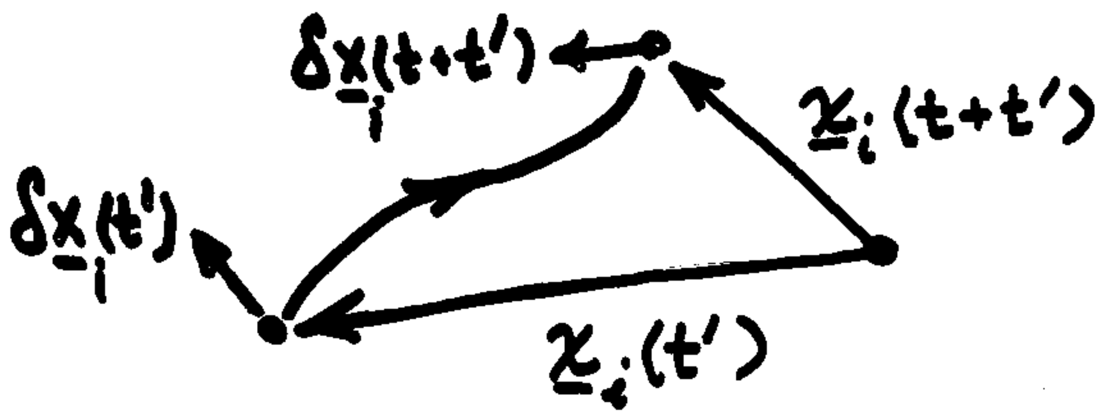
"Incoherent State": $\langle\langle \underline{x} \rangle\rangle = \langle\langle \underline{x} \rangle\rangle_*$

STABILITY OF THE INCOHERENT STATE

Goal: Obtain stability of coupled system from dynamics of the uncoupled components

- $\underline{x}_i(t) \rightarrow \underline{x}_i(t) + \delta\underline{x}_i(t)$
- $d\delta\underline{x}_i/dt = \underline{D}\underline{G}\left(\underline{x}_i(t), \underline{\Omega}_i\right)\delta\underline{x}_i - \underline{\underline{K}}\,\langle\langle\delta\underline{x}$

Fundamental (Lyapunov) matrix $\underline{\underline{M}}_i$ of the uncoupled system:



$$\delta\underline{x}_i(t+t') = \underline{\underline{M}}_i\left(t, t'; \underline{x}(t')\right)\delta\underline{x}(t')$$

$$\underline{\underline{M}}_i\left(0, t'; \underline{x}(t')\right) \equiv \underline{\underline{1}}$$

$$d\underline{\underline{M}}_i/dt = \underline{D}\,\underline{G}\,(\underline{x}_i(t+t'), \underline{\Omega}_i)\,\underline{\underline{M}}_i$$

$$\underline{\underline{M}}_i\,(0, t'; \underline{x}\,(t')) = \underline{\underline{1}}$$

<u>Solution for $\delta\underline{x}_i$:</u>

$$\delta\underline{x}_i = -\int_{-\infty}^{t} \underline{\underline{M}}_i(t-\tau, \tau; \underline{x}_i(\tau))\,\underline{\underline{K}}\,\langle\langle\delta\underline{x}\rangle\rangle_\tau\, d\tau$$

<u>Dispersion function :</u>

Take $\langle\langle\ldots\rangle\rangle_*$, assume $\langle\langle\delta\underline{x}\rangle\rangle_* = \underline{\Delta}\,e^{st}$,

and let $T = t - \tau$:

$$\{\,\underline{\underline{1}} + \underline{\underline{\tilde{M}}}(s)\,\underline{\underline{K}}\,\}\cdot\underline{\Delta} = 0 \;\Rightarrow\; D(s) = \det\{\ldots\} = 0$$

where

$$\underline{\underline{\tilde{M}}}(s) = \left\langle\!\left\langle \int_0^\infty e^{-sT}\,\underline{\underline{M}}_i(T, \ldots)\,dT \right\rangle\!\right\rangle_*$$

$$\boxed{\underline{\underline{\tilde{M}}}(s)}$$

$$\underline{\underline{\tilde{M}}}(s) = \left\langle\!\left\langle \int_0^\infty e^{-sT}\, \underline{\underline{M}}_i(T)\, dT \right\rangle\!\right\rangle_*$$

Convergence

- $\mathrm{Re}(s) > \Gamma \equiv \max\limits_{\underline{x}_i, \underline{\Omega}_i} h_i$

$h_i(\underline{x}_i, \underline{\Omega}_i)$ = largest Lyap. exponent

$h_i = 0$ for limit cycles

h_i and $\Gamma > 0$ for chaotic attractors

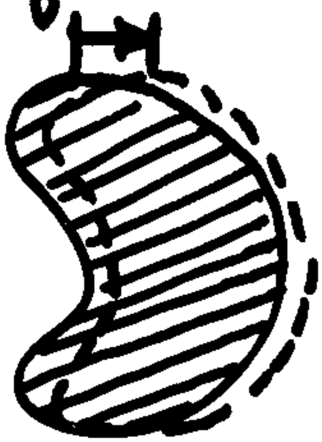
<u>For $\mathrm{Re}(s) > \Gamma$:</u>

$$\underline{\underline{\tilde{M}}}(s) = \int_0^\infty e^{-sT} \left\langle\!\left\langle \underline{\underline{M}}_i(T) \right\rangle\!\right\rangle_* dT$$

(Not yet useful.)

DECAY OF $\langle\langle \underline{M}_i(T) \rangle\rangle_*$ ←

Mixing Chaotic Attractors



$\underline{x}_i(0) \to \underline{x}_i(0) + \delta_k \underline{a}_k$

$\underline{a}_k$ = unit vector in k direction

→ $[\langle\langle \underline{M}_i \rangle\rangle_*]_k \delta_k$ = perturbation of centroid of cloud ~~in direction k~~

k^{th} column

→ mixing ⇒ perturbation decays to zero. (Typically exponentially.)

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Limit Cycle Attractors

- Not mixing

$\langle \underline{\underline{M}}_i \rangle_* =$ average over invariant measure μ_i.

- Example:

$$\langle \underline{\underline{M}}_i \rangle_* = \frac{1}{2} \begin{bmatrix} \cos \Omega_i t & -\sin \Omega_i t \\ \sin \Omega_i t & \cos \Omega_i t \end{bmatrix}$$

- Average over $\rho(\Omega)$ to get $\langle\langle \underline{\underline{M}} \rangle\rangle_*$:

$$\int \rho(\Omega) \left\{ \begin{matrix} \cos \Omega t \\ \sin \Omega t \end{matrix} \right\} d\Omega \sim e^{-\gamma t}$$

$\gamma > 0$ for $\rho(\Omega)$ analytic.

ANALYTIC CONTINUATION

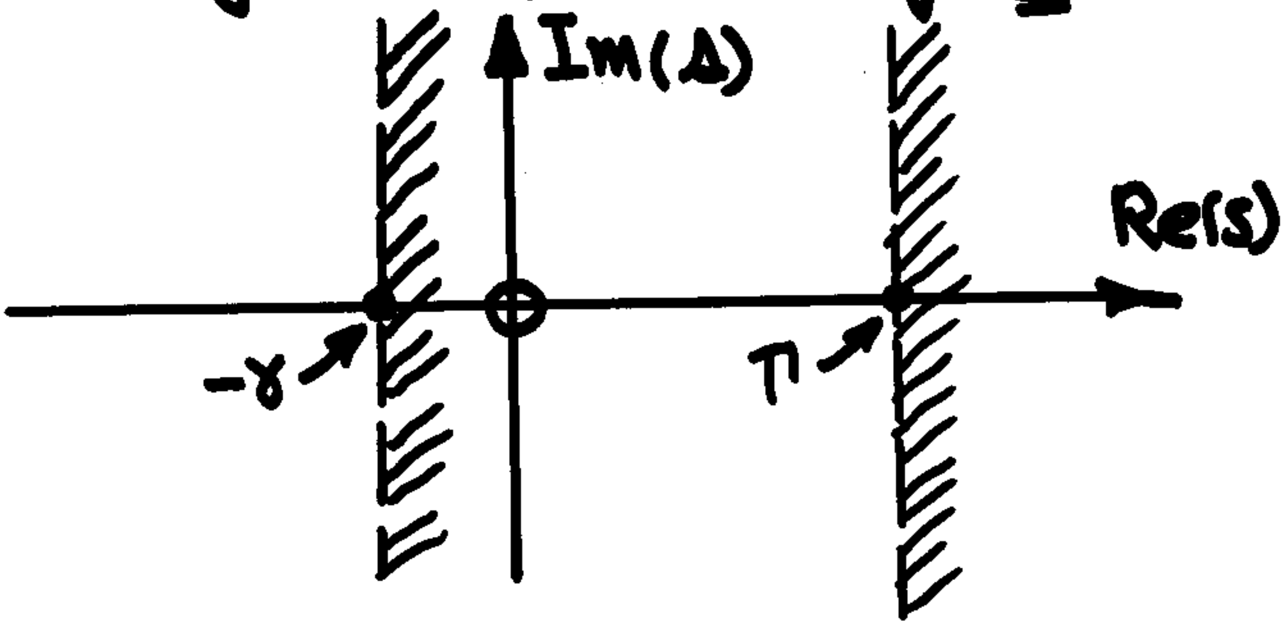
$$\tilde{\underline{\underline{M}}}(s) = \left\langle\!\!\left\langle \int_0^\infty e^{-sT}\, \underline{\underline{M}}_i(T)\, dT \right\rangle\!\!\right\rangle_*$$

$$= \int_0^\infty e^{-sT} \left\langle\!\!\left\langle \underline{\underline{M}}_i(T) \right\rangle\!\!\right\rangle_* dT \quad \text{for } \mathrm{Re}(s) > \Gamma \geq 0$$

- Reasonable assumption

$$\left\| \left\langle\!\!\left\langle \underline{\underline{M}}(t) \right\rangle\!\!\right\rangle_* \right\| < K e^{-\gamma t}, \; \gamma > 0$$

$\Rightarrow$ Analytic continuation of $\tilde{\underline{\underline{M}}}(s)$:



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EXAMPLE: A KURAMOTO-TYPE MODEL

$$\underline{x}_i(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \qquad 2D$$

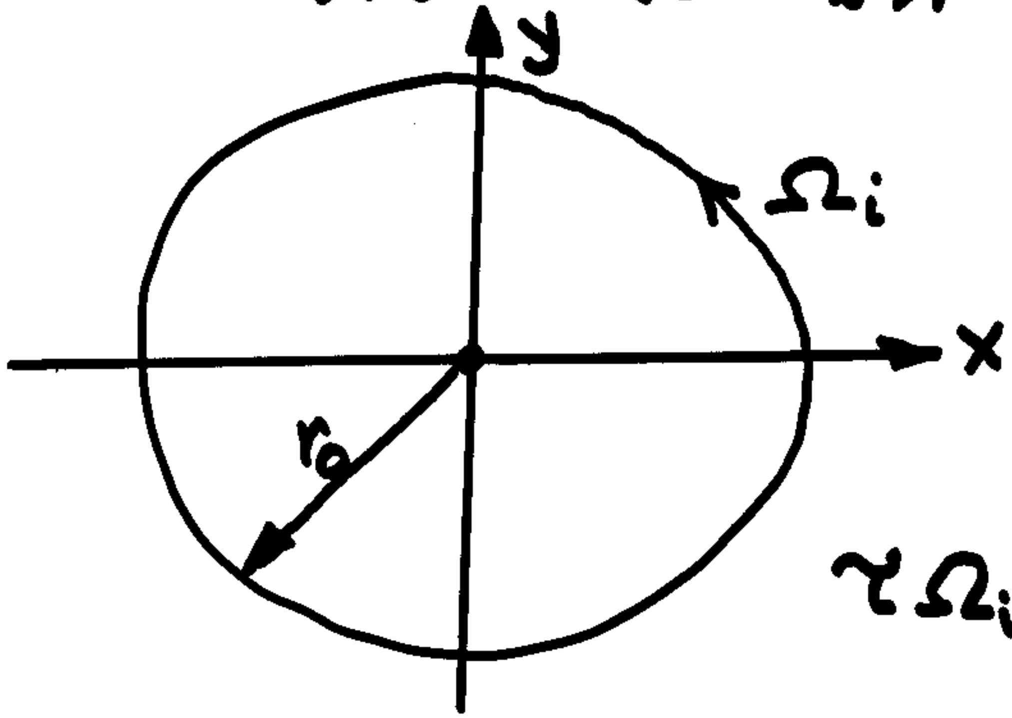
$$dx_i/dt = G^{(x)}(x_i, y_i, \Omega_i) + k(\langle\langle x \rangle\rangle_* - \langle\langle x \rangle\rangle)$$

$$dy_i/dt = G^{(y)}(x_i, y_i, \Omega_i) + k(\langle\langle y \rangle\rangle_* - \langle\langle y \rangle\rangle)$$

In polar coordinates with $k = 0$

$$d\theta_i/dt = \Omega_i$$

$$dr_i/dt = (r_0 - r_i)/\tau$$



$\tau \Omega_i \ll 1$

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STABILITY FOR KURAMOTO MODEL

- Approximation: $\tau\Omega_i \ll 1 \quad t \gg \tau$

- Infinitesmal orbit perturbation at $t=0$:

$$\underline{\Delta}_{oi} = \underline{a}_x \, dx_{oi} + \underline{a}_y \, dy_{oi}$$

- At time t such that $\frac{2\pi}{\Omega} \gg t \gg \tau$

$$r \to r_o \,, \quad \theta \cong \theta_{oi}$$

$$\underline{\Delta}_i(t) \cong \Delta^+_{oi} \, \underline{a}_\theta$$ evaluated at $\theta = \theta_{oi}$

- Subsequently

$$\underline{\Delta}_i(t) = \Delta^+_{oi} \, \underline{a}_\theta$$ evaluated at $\theta = \theta_{oi} + \Omega_i t$

- Can evaluate $\underline{\underline{M}}_i$ from def.:

$$\begin{bmatrix} dx_i(t) \\ dy_i(t) \end{bmatrix} = \underline{\underline{M}}_i \begin{bmatrix} dx_{oi} \\ dy_{oi} \end{bmatrix}$$

- Average over θ_{oi}: $\Rightarrow \langle \underline{\underline{M}}_i \rangle_* = \frac{1}{2} \begin{bmatrix} \cos\Omega_i t \ldots \\ \sin\Omega_i t \ldots \end{bmatrix}$

- Dispersion Relation: $$1 + \frac{k}{2} \int_{-\infty}^{+\infty} \frac{\rho(\Omega)\, d\Omega}{s \pm i\Omega} = 0$$

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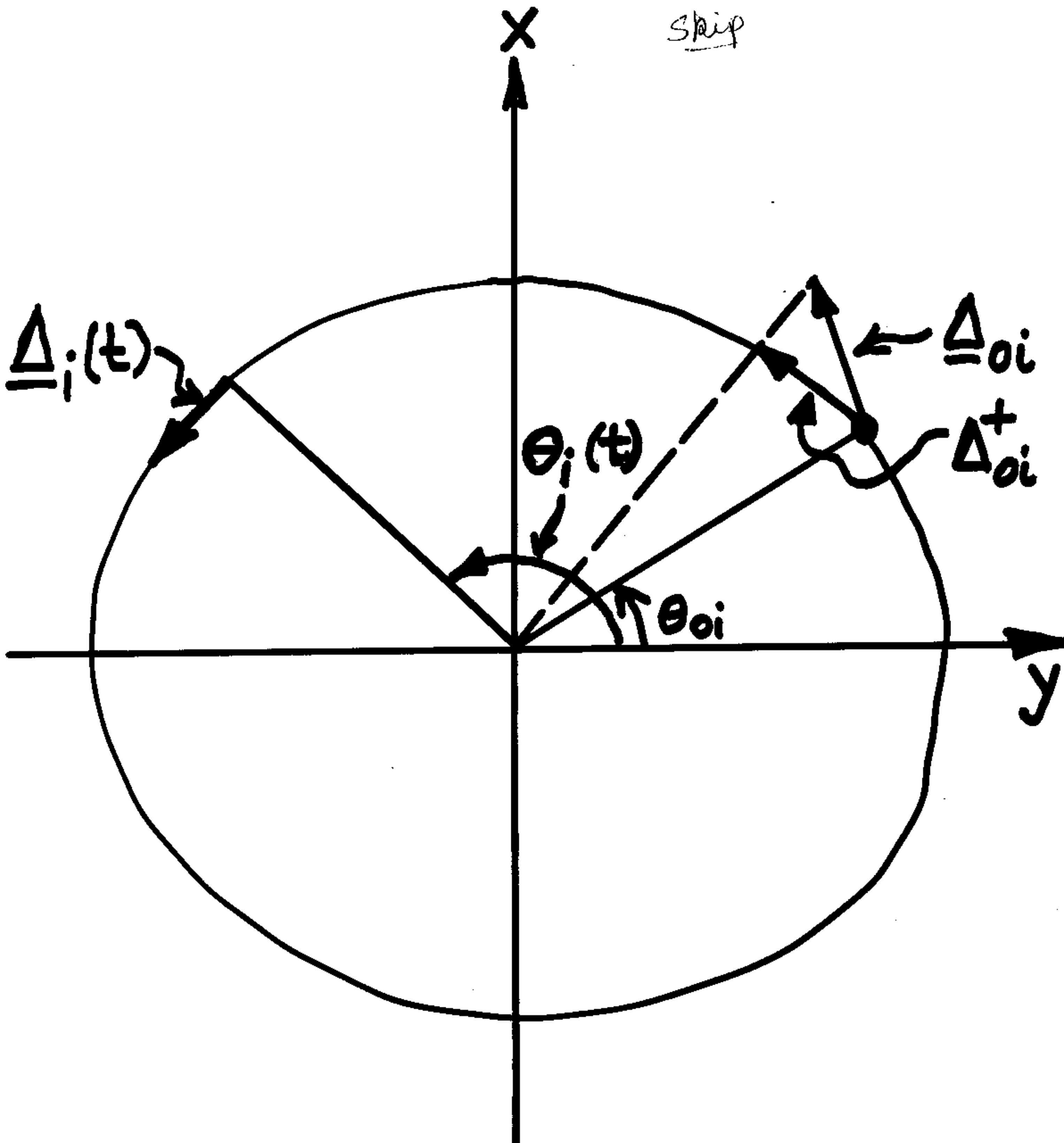


$$\theta_i(t) = \theta_{oi} + \Omega_i t$$

$$\underline{\Delta}_{oi} = \begin{pmatrix} dx_{oi} \\ dy_{oi} \end{pmatrix} \qquad \Delta_i(t) = \begin{pmatrix} dx_i(t) \\ dy_i(t) \end{pmatrix}$$

Numerical Implimentation

x-equation of Lorentz ensemble:

$$dx_i/dt = \sigma(y_i - x_i) - k\,\langle\langle x \rangle\rangle$$

Dispersion function:

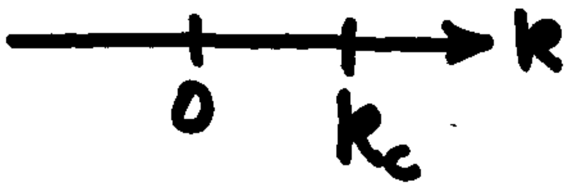
$$D(s) = 1 \mp k\,\tilde{M}_{11}(s) = 0$$

Marginal stability: $s = -i\omega$

$$1 \mp k\,\tilde{M}_{11}(-i\omega) = 0$$

$$\mathrm{Im}\,\tilde{M}_{11}(-i\omega) = 0 \quad \Rightarrow \omega = \omega_*$$

$$k_c = {}^{\pm 1}/\,\tilde{M}_{11}(-i\omega_*)$$



Perturb around k_c: $k = k_c + \delta k$

$$s = -i(\omega_* + \delta\omega) + \gamma \quad (\gamma, \delta\omega, \delta k \text{ small})$$

$$\gamma \doteq -\frac{\delta k}{(k_c)^2}\,\frac{\mathrm{Im}\ \partial\tilde{M}_{11}/\partial\omega}{|\partial\tilde{M}_{11}/\partial\omega|^2}$$

Methods for Evaluating $\tilde{M}_{11}(-i\omega)$

- Displace orbits in the ensemble by a small amount. Look at relaxation of centroid back to its equilibrium position. Take Laplace (or Fourier) transform.

- $\tilde{M}_{11}(-i\omega)$ is
  (response to $\Delta e^{-i\omega t}$) $\div$ $(\Delta e^{-i\omega t})$

$$dx_i^{c,s}/dt = \sigma(y_i^{c,s} - x_i^{c,s}) + \Delta \begin{cases} \cos \omega t \\ \sin \omega t \end{cases}$$

$$\tilde{M}_{11}(-i\omega) \cong \Delta^{-1} e^{i\omega t} \langle\langle x_i^c - i\, x_i^s \rangle\rangle$$

- For hyperbolic chaotic attractors the measure can be expressed via an average over unstable periodic orbits embedded in the attractor.

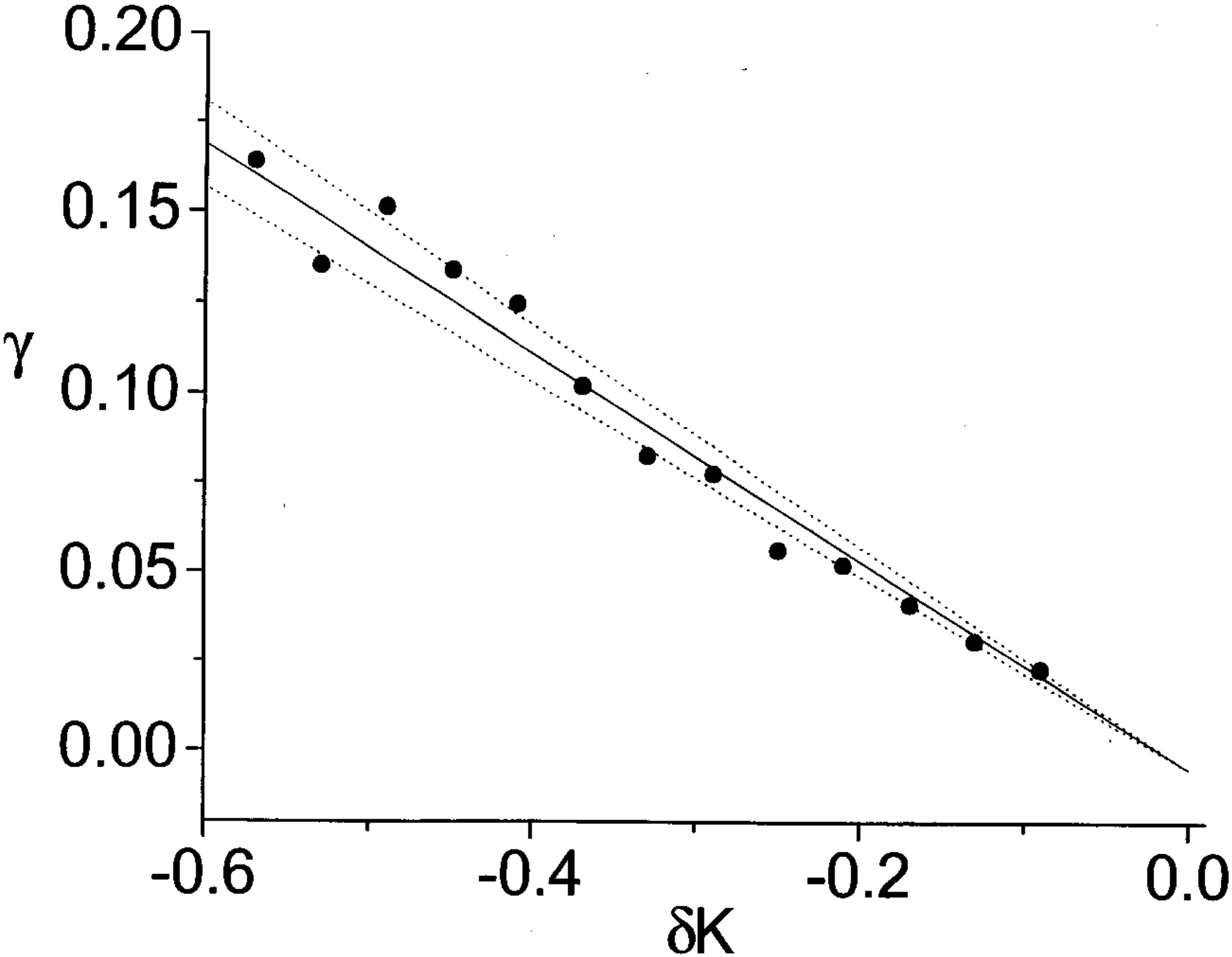


Fig. 1c

CONCLUSION

- Useful formalism for the study of globally coupled ensembles of dynamical systems.

- Applies to periodic, chaotic and "mixed" ensembles.