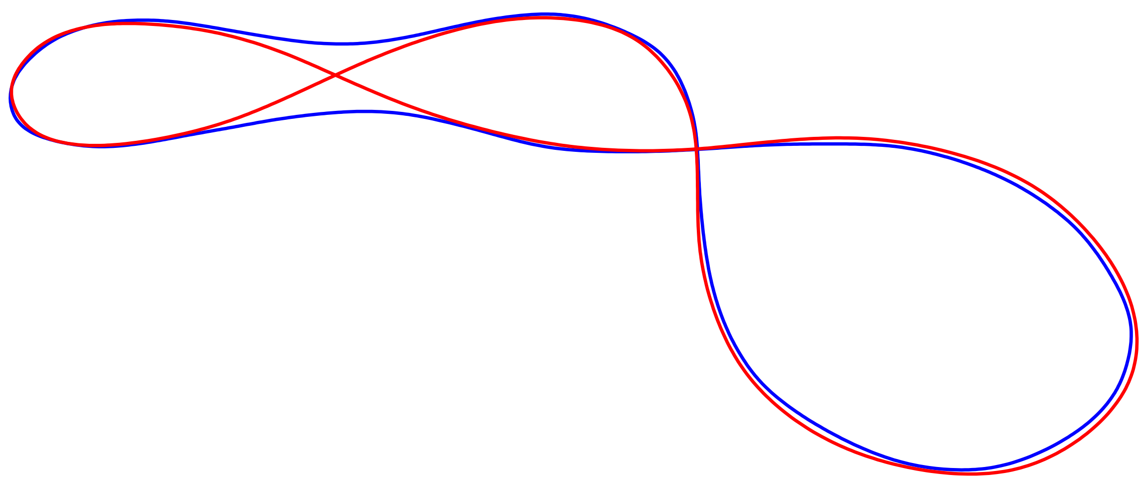
# Semiclassical evidence for universal spectral correlations

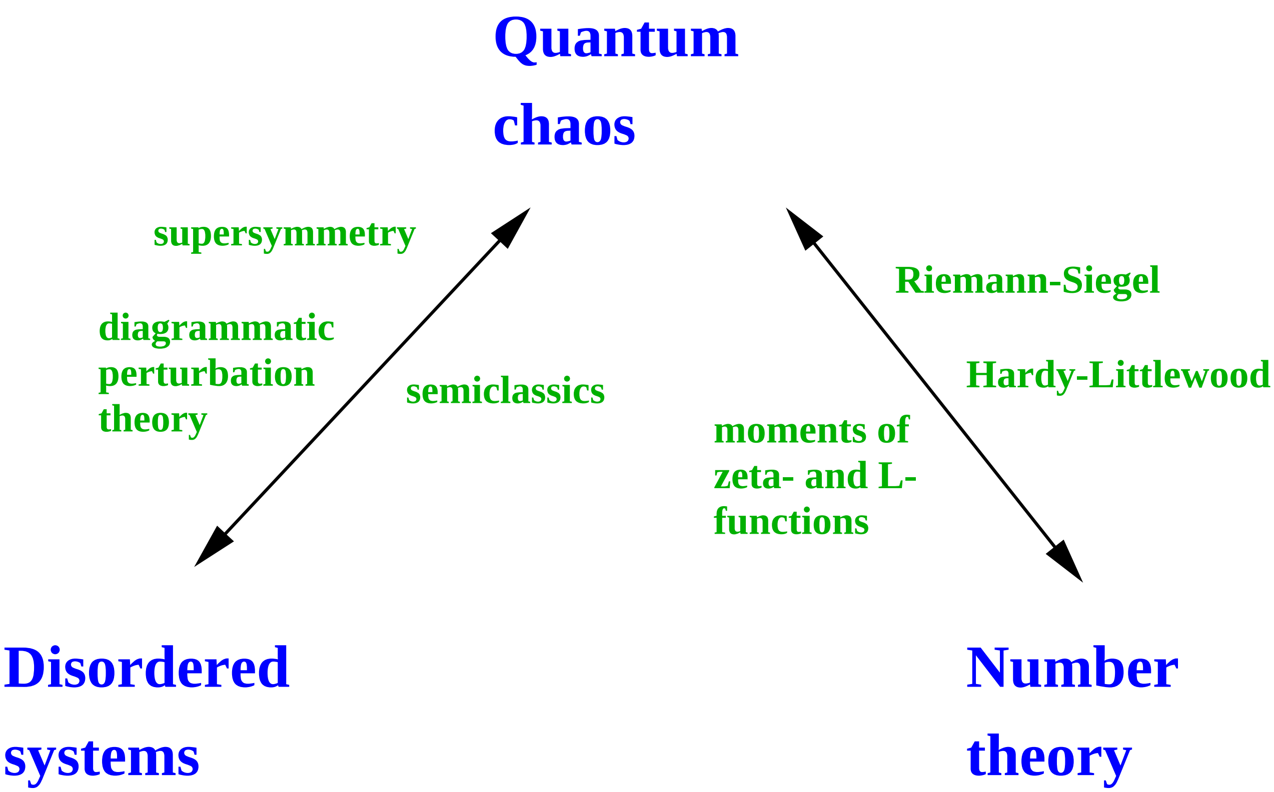
Martin Sieber

University of Bristol



Classical basis for universality in quantum spectra

# RMT applications




Newton Institute programmes:

- Disordered systems and quantum chaos (1997)
- Random matrix approaches in number theory (2004)

# Spectral form factor

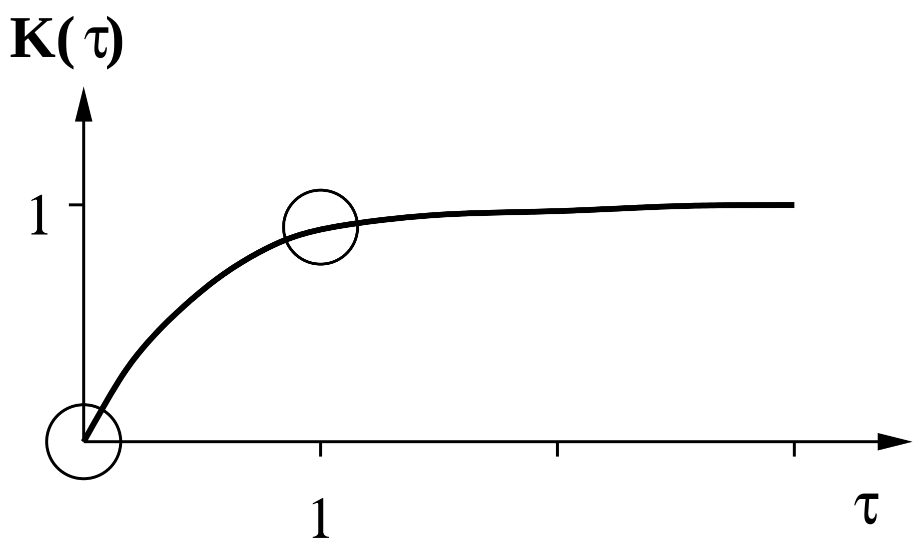
The spectral form factor

$$K(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\eta}{\bar{d}(E)} \left\langle d_{\mathrm{osc}}\left(E + \frac{\eta}{2}\right) d_{\mathrm{osc}}\left(E - \frac{\eta}{2}\right) \right\rangle_{E,\tau} \exp\{2\pi i \eta \tau \bar{d}(E)\}$$

where $d(E) = \sum_n \delta(E - E_n)$, and $d_{\mathrm{osc}}(E) = d(E) - \bar{d}(E)$.

Chaotic systems with time reversal symmetry: agreement with the form factor of the Gaussian Orthogonal Ensemble as $E \to \infty$.

$$K^{\mathrm{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1 + 2\tau) & \text{if } \tau < 1 \\ 2 - \tau \log \frac{2\tau+1}{2\tau-1} & \text{if } \tau > 1 \end{cases}$$




Expansion for small values of $\tau$

$$K^{\mathrm{GOE}}(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \ldots$$

# Gutzwiller trace formula

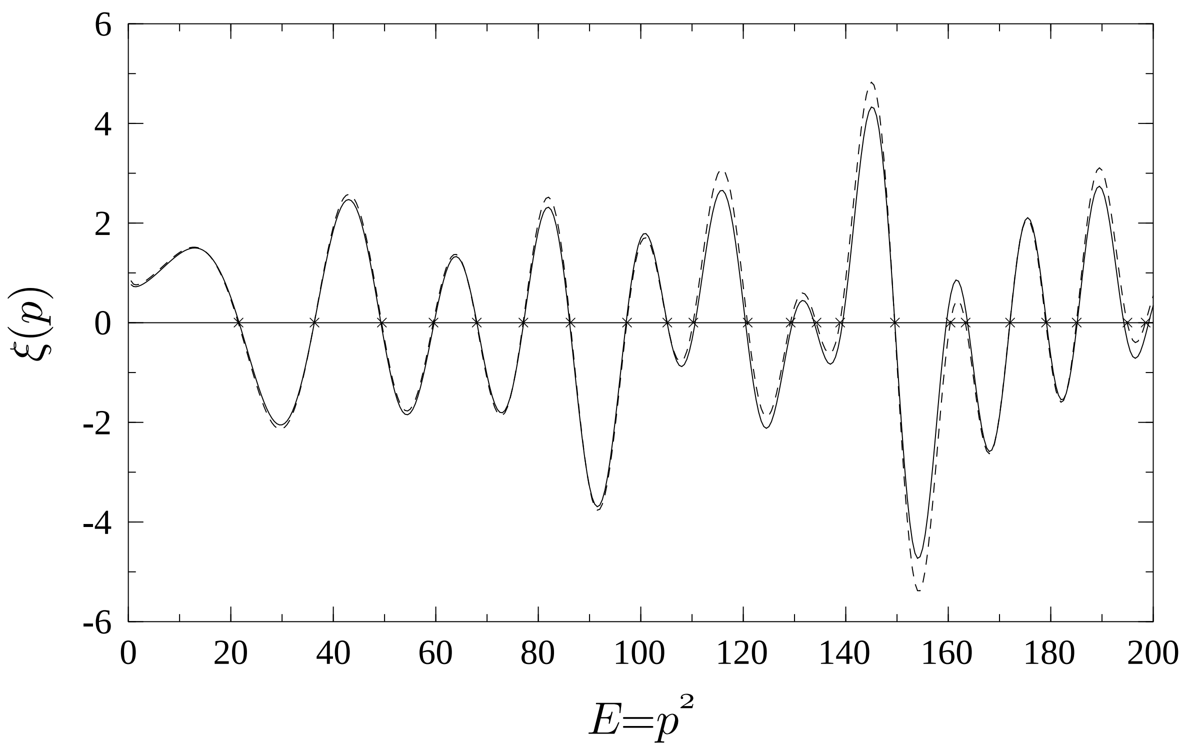
Trace formula for the density of states

$$d(E) = \sum_n \delta(E - E_n) \approx \bar{d}(E) + \frac{1}{\pi\hbar} \mathrm{Re} \sum_\gamma A_\gamma e^{iS_\gamma(E)/\hbar}$$

where $\bar{d}(E) \sim \Sigma(E)/h^d$. Gutzwiller (1971);

$$S_\gamma(E) = \int_\gamma p \, dq$$

Trace formula is exact for Riemann surfaces of constant negative curvature $\Longrightarrow$ Selberg trace formula.



# The diagonal approximation

Semiclassical approximation for $K(\tau)$

$$K(\tau) \approx \frac{1}{T_H} \sum_{\gamma,\gamma'} \left\langle A_\gamma A^*_{\gamma'}\, e^{i(S_\gamma - S_{\gamma'})/\hbar}\, \delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right)\right\rangle$$

where $T_H = 2\pi\hbar\bar{d}(E)$ and $\tau = T/T_H$

Diagonal approximation (Berry, 1985)

$$K_{\mathsf{d}}(\tau) \approx \frac{2}{T_H} \sum_{\gamma} |A_\gamma|^2\, \delta(T - T_\gamma)$$

Sum rule, Hannay, Ozorio de Almeida (1984)

$$\sum_{\gamma} |A_\gamma|^2\, \delta(T - T_\gamma) \sim T$$

Result:

$$K_{\mathsf{d}}(\tau) \approx 2\tau$$

Interpretation, Argaman et al.(1993): Probability to return $P(T)$

$$K_{\mathsf{d}}(\tau) = 4\pi\hbar T\, P(T)$$

# Action correlations

Universality in energy statistics $\Longleftrightarrow$ universality in statistics of periodic orbits

Orbits have to be correlated! Define

$$P(\Delta S, T) = \left\langle \sum_{\gamma \neq \gamma'} A_\gamma A^*_{\gamma'} \, \delta\left(\Delta S - (S_\gamma - S_{\gamma'})\right) \, \delta\left(T - \frac{T_\gamma + T_{\gamma'}}{2}\right) \right\rangle$$

$$P(\Delta S, T) \sim \frac{2\pi T^2}{\Sigma(E)} \, \hat{P}_{\mathsf{GOE}} \left( \frac{2\pi T \Delta S}{\Sigma(E)} \right) + F(T) \quad \text{as} \quad T \to \infty$$

where

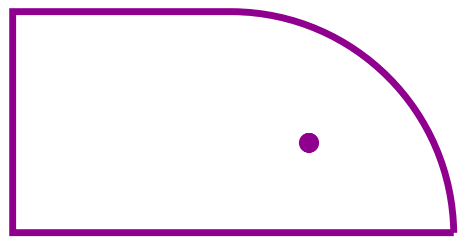
$$P_{\mathsf{GOE}}(y) = -\frac{4}{\pi} \left[ \frac{\sin(y/2)}{y} \right]^2 + \frac{2}{\pi y} \, \left[ -\cos y \, f(y) + f(2y) \right]$$

and $f(y) = \mathrm{Ci}(y) \;\; \sin(y) - \mathrm{si}(y) \;\; \cos(y)$. (Argaman et al., 1993)

Limiting behaviour: Action repulsion!

$$P_{\mathsf{GOE}} \sim \frac{2}{\pi} \, \log y \qquad \text{as} \qquad y \to 0$$

# Perturbation by a point scatterer



Addition of a point-like scatterer to a system

$$\hat{H} = \hat{H}_0 + \text{``}\alpha\delta(\boldsymbol{r} - \boldsymbol{r}_0)\text{''}$$

Not defined for $d \geq 2$. Apply method of self-adjoint extension.

Energy spectrum of the new system

$$1 - \alpha \sum_{n=1}^{\infty} \left[ \frac{|\psi_n(\boldsymbol{r}_0)|^2}{E - E_n} - \frac{|\psi_n(\boldsymbol{r}_0)|^2}{E' - E_n} \right] = 0$$

Unperturbed energies and wave functions at $\boldsymbol{r}_0$ determine the new energy spectrum.

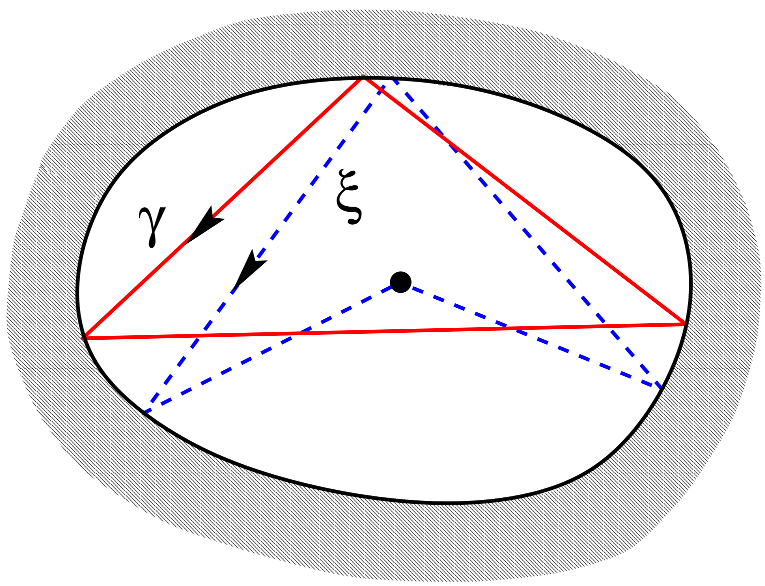
Point-like scatterer does not change the spectral statistics in generic chaotic system. (Stöckmann, Seba (1998), Bogomolny, Leboeuf and Schmidt (2000)).

# Diffractive orbits

Trace formula for the density of states

$$d(E) \approx \bar{d}(E) + \frac{1}{\pi\hbar} \mathrm{Re} \sum_{n=0}^{\infty} \sum_{\gamma} A_{\gamma}^{(n)} e^{iS_{\gamma}^{(n)}(E)/\hbar}$$

Vattay, Wirzba, Rosenqvist (1994). The sum runs over all periodic orbits and all diffractive orbits. For $n$-fold diffractive orbits $A_{\gamma}^{(n)} = \mathcal{O}(\hbar^{n/2})$.



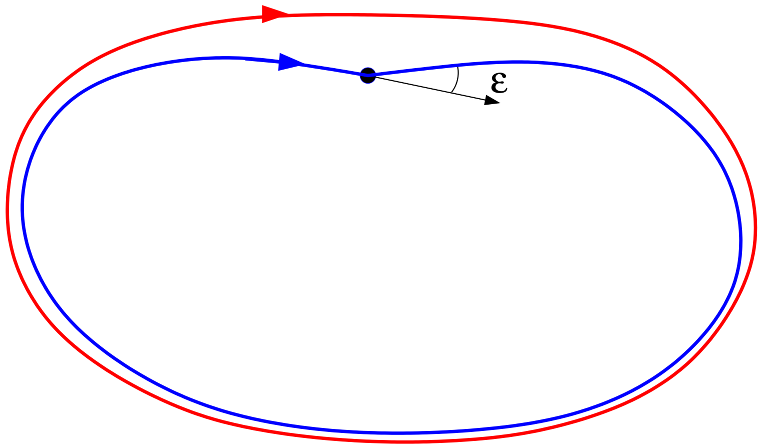

Diagonal approximation for diffractive orbits

$$K_d^{(n)}(\tau) = \frac{|\mathcal{D}|^{2n}\ \tau^{n+1}}{2^n\ n}$$

where $\mathcal{D}$ is the diffraction coefficient that describes the scattering process at the point scatterer. (M.S. 1999)

# First-order terms

Correlations between single-diffractive orbits and periodic orbits




The off-diagonal contribution to the form factor is obtained by summing over all diffractive orbits which are almost periodic

$$K_{\text{off}}^{(1)}(\tau) = \frac{4}{T_H}\,\text{Re}\int_{-\infty}^{\infty} \mathrm{d}\varepsilon \sum_{\gamma}^{(\varepsilon)} A_{\gamma}^{\text{1do}} (A_{\gamma}^{\text{po}})^{*}\,\delta(T - T_{\gamma}) \times \exp\left(-\frac{i}{\hbar}\Delta S_{\gamma}(E)\right)$$

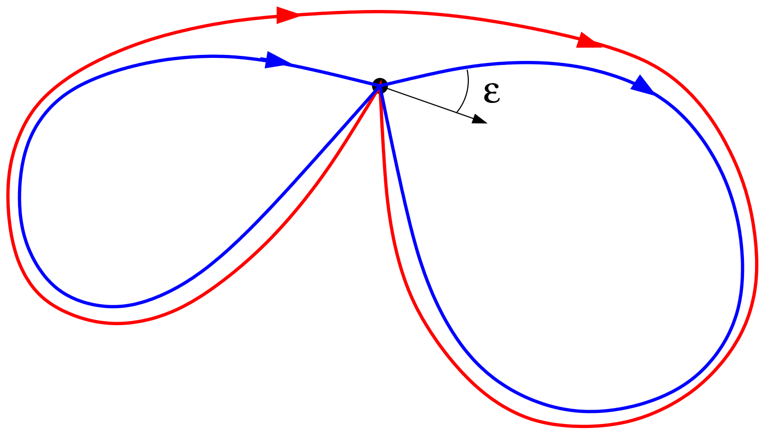
The joint result is

$$K_{\text{d}}^{(1)}(\tau) + K_{\text{off}}^{(1)}(\tau) = \frac{|\mathcal{D}|^2}{2\beta}\tau^2 + \frac{2}{\beta}\tau^2\,\text{Im}\,D = 0$$

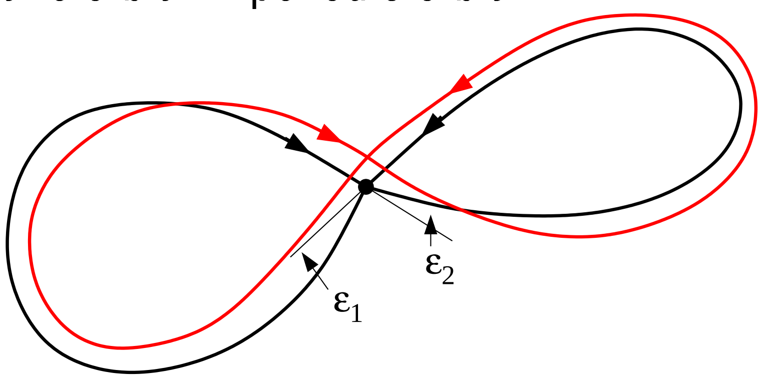
where we used $|\mathcal{D}|^2 = -4\,\text{Im}\,D$ (optical theorem). Bogomolny, Leboeuf and Schmit (2000), M.S. (2000).

# Second-order terms (GUE)

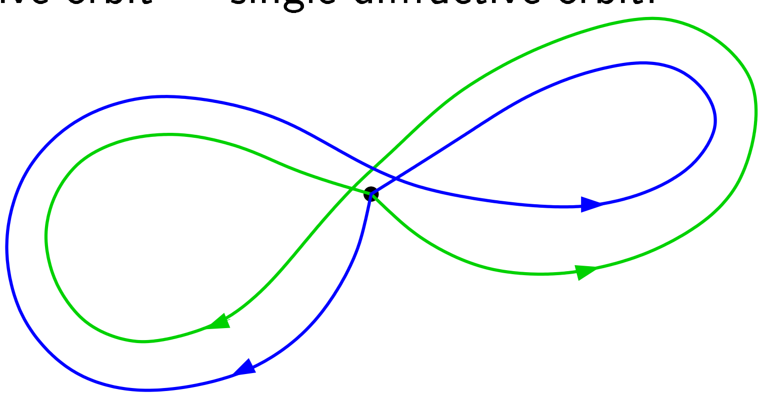
double-diffractive orbit — single-diffractive orbit:




double-diffractive orbit — periodic orbit:



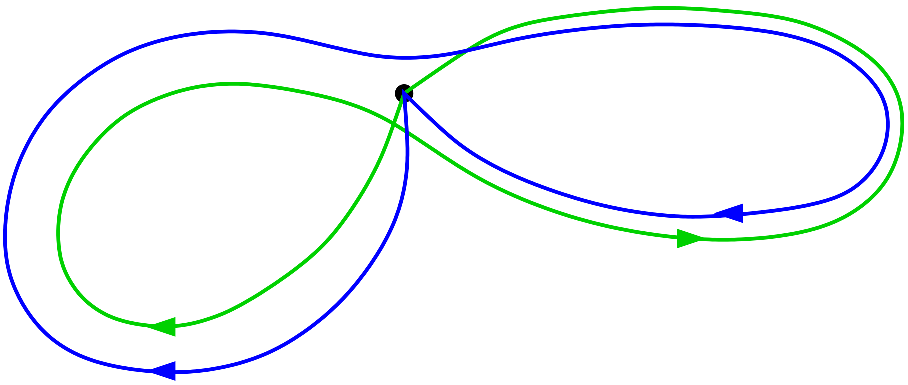

single-diffractive orbit — single-diffractive orbit:



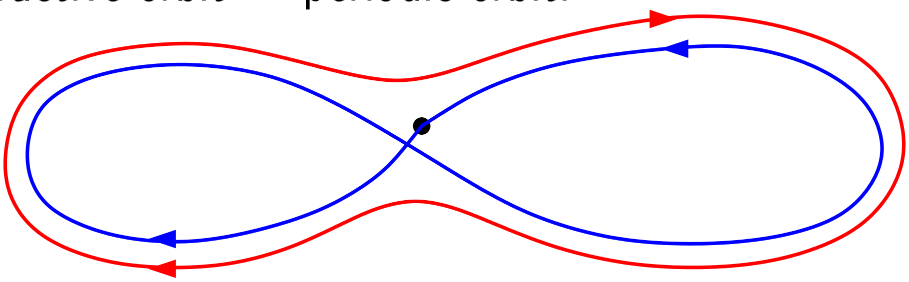
$$K^{(2)}(\tau) = \frac{\tau^3}{4}\left(\frac{1}{8}|\mathcal{D}|^4 + |\mathcal{D}|^2 \operatorname{Im}\mathcal{D} - \operatorname{Re}\mathcal{D}^2 + |\mathcal{D}|^2\right) = 0$$

# Additional terms (GOE)

single-diffractive orbit — single-diffractive orbit:



single-diffractive orbit — periodic orbit:



$$K^{(2)}(\tau) = \tau^3 \left( \frac{1}{8}|\mathcal{D}|^4 + |\mathcal{D}|^2 \operatorname{Im} \mathcal{D} - \operatorname{Re} \mathcal{D}^2 + \frac{1}{2}|\mathcal{D}|^2 + |\mathcal{D}|^2 + 2 \operatorname{Im} \mathcal{D} \right) = 0$$

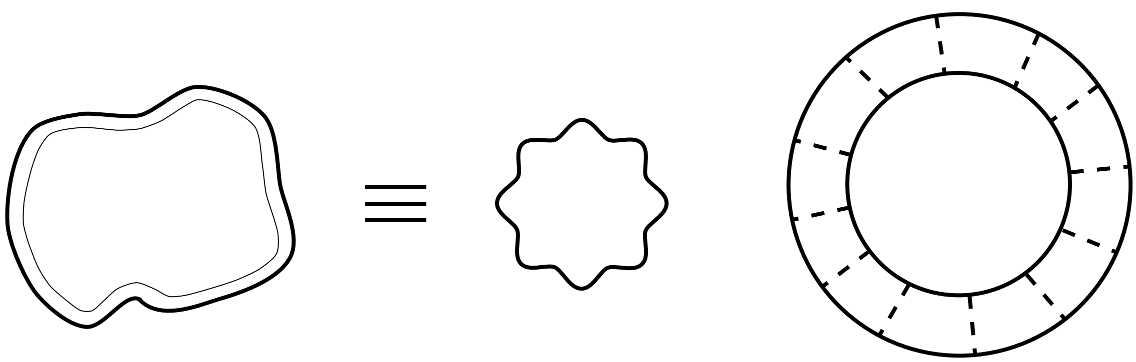
# Disordered systems

Diagrammatic perturbation expansion for the first three terms of the spectral form factor

$$K^{\mathsf{GOE}}(\tau) = 2\tau - 2\tau^2 + 2\tau^3 + \dots$$

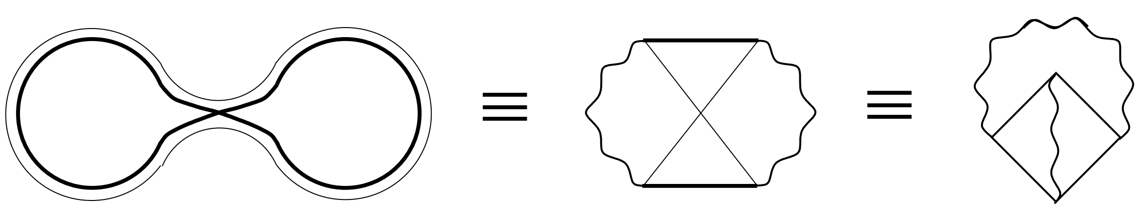
Smith, Lerner and Altshuler (1998)

One-loop diagram

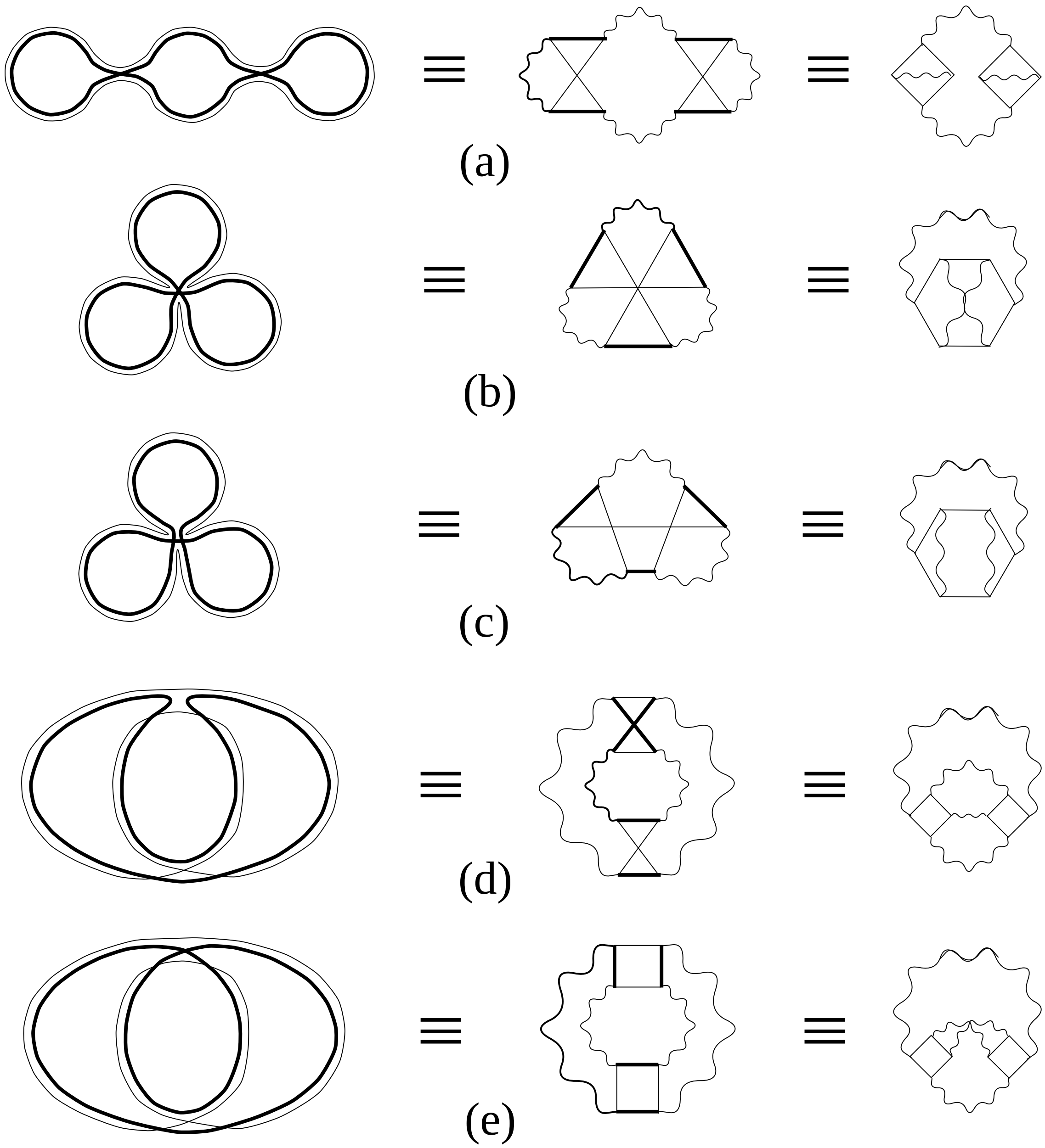


(a) (b)

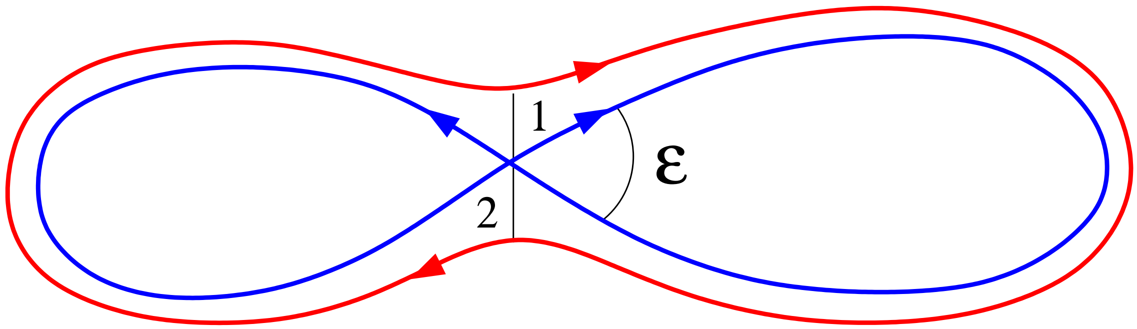
Two-loop diagram

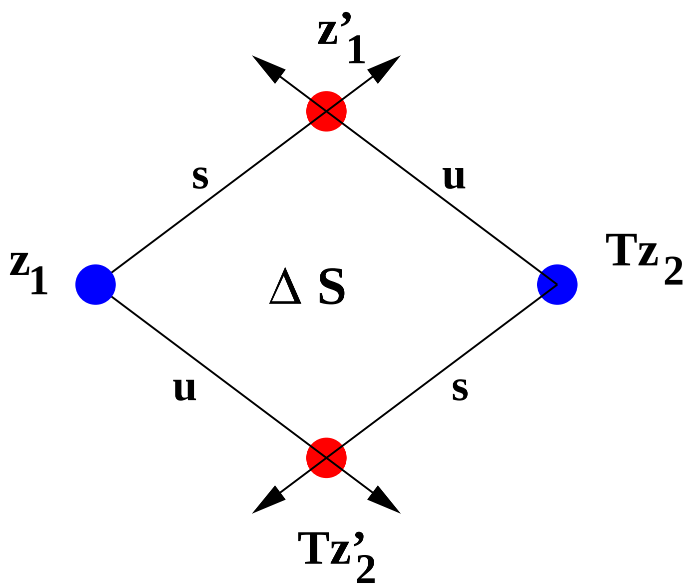


# Three-loop diagrams



# Action difference




Action difference

$$\Delta S = u\ s = \frac{p^2 \varepsilon^2}{2m\lambda}$$

where $|\vec{e}_s \times \vec{e}_u| = 1.$

(Turek, Richter (2003), Spehner (2003), Müller (2003))

# Number of loops

Uniformly hyperbolic systems:

Average number of loops along trajectory of length $T$ characterised by $u = -s$ in $[u, u + du]$ is given by $P(u, T)\, du$ where

$$P(u,T) \sim \frac{2\lambda T^2}{\Sigma(E)} |u| \quad \text{as} \quad T \to \infty$$

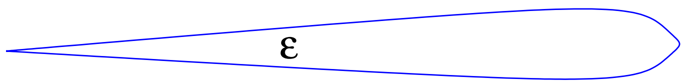
Contribution to the form factor

$$K^{(2)}_{\text{off}}(\tau) = \frac{4}{T_H} \operatorname{Re} \int du \sum_{\gamma} A_{\gamma}^2\, P(u,T)\, e^{i\Delta S/\hbar}\, \delta(T - T_{\gamma})$$
$$\approx 0$$

The $\tau^2$-term of the form factor originates from the next-order correction to the large $T$ asymptotics of $P(u, T)$

# Correction to loop distribution

Minimal loop length estimate: $u \exp(\lambda T_{\mathsf{min}}/2) = \mathcal{O}(1)$




Periodic orbit expansion of $P(u,T)$ involving

$$|u| \ \cosh \frac{\lambda T}{2} = \cosh \frac{\lambda T_0}{2}$$

yields correction to the loop distribution

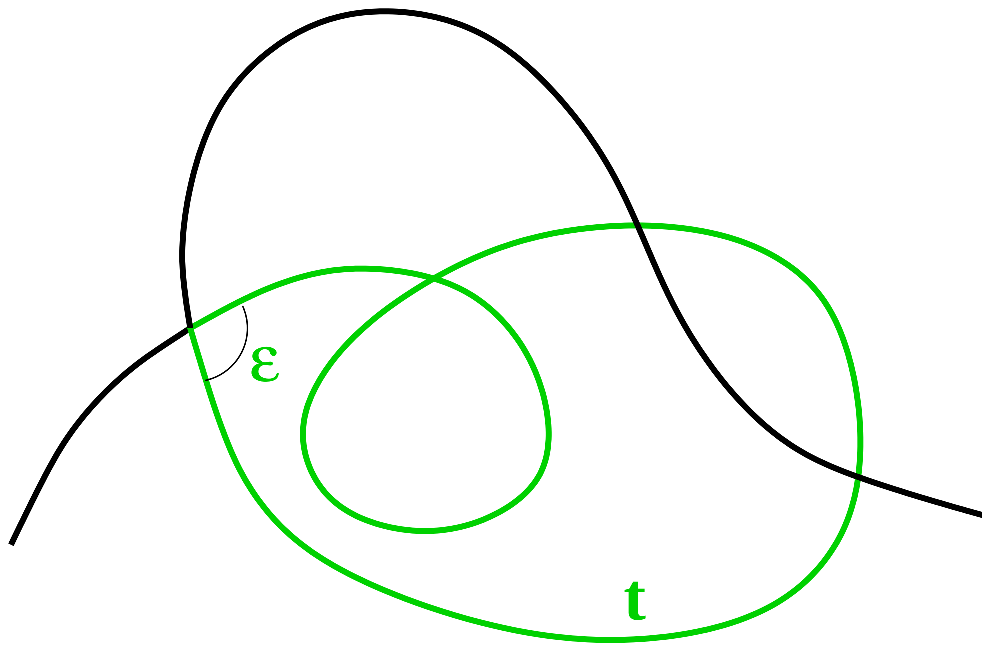
$$P(u,T) \sim \frac{2\lambda T^2}{\Sigma(E)} |u| \ \left(1 + \frac{4}{\lambda T} \log(cu)\right)$$

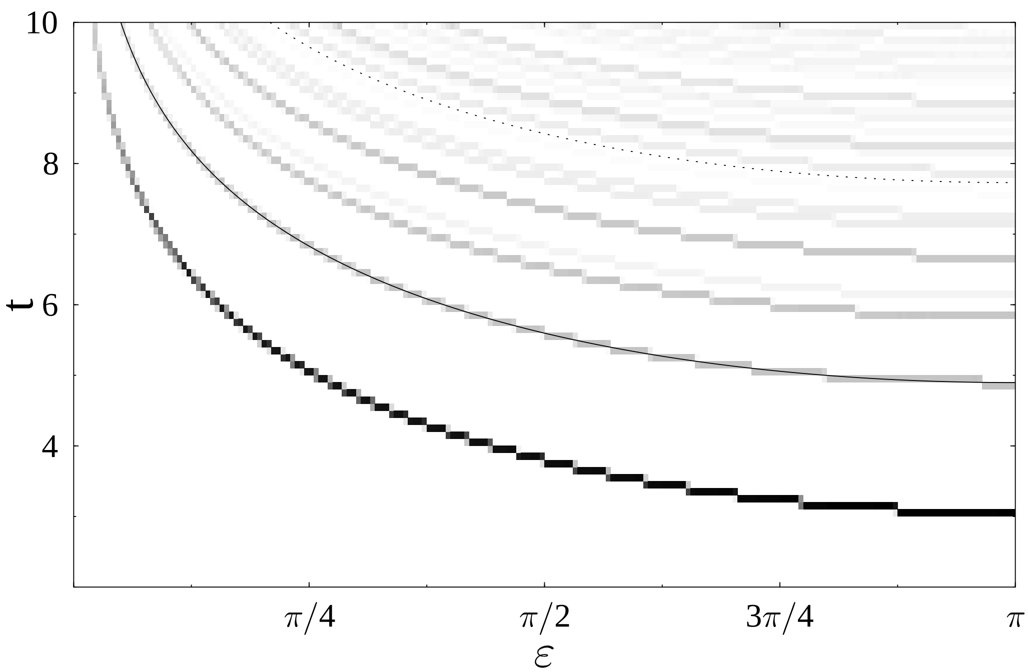
Result for formfactor

$$K_{\mathsf{off}}^{(2)}(\tau) = 2\tau$$

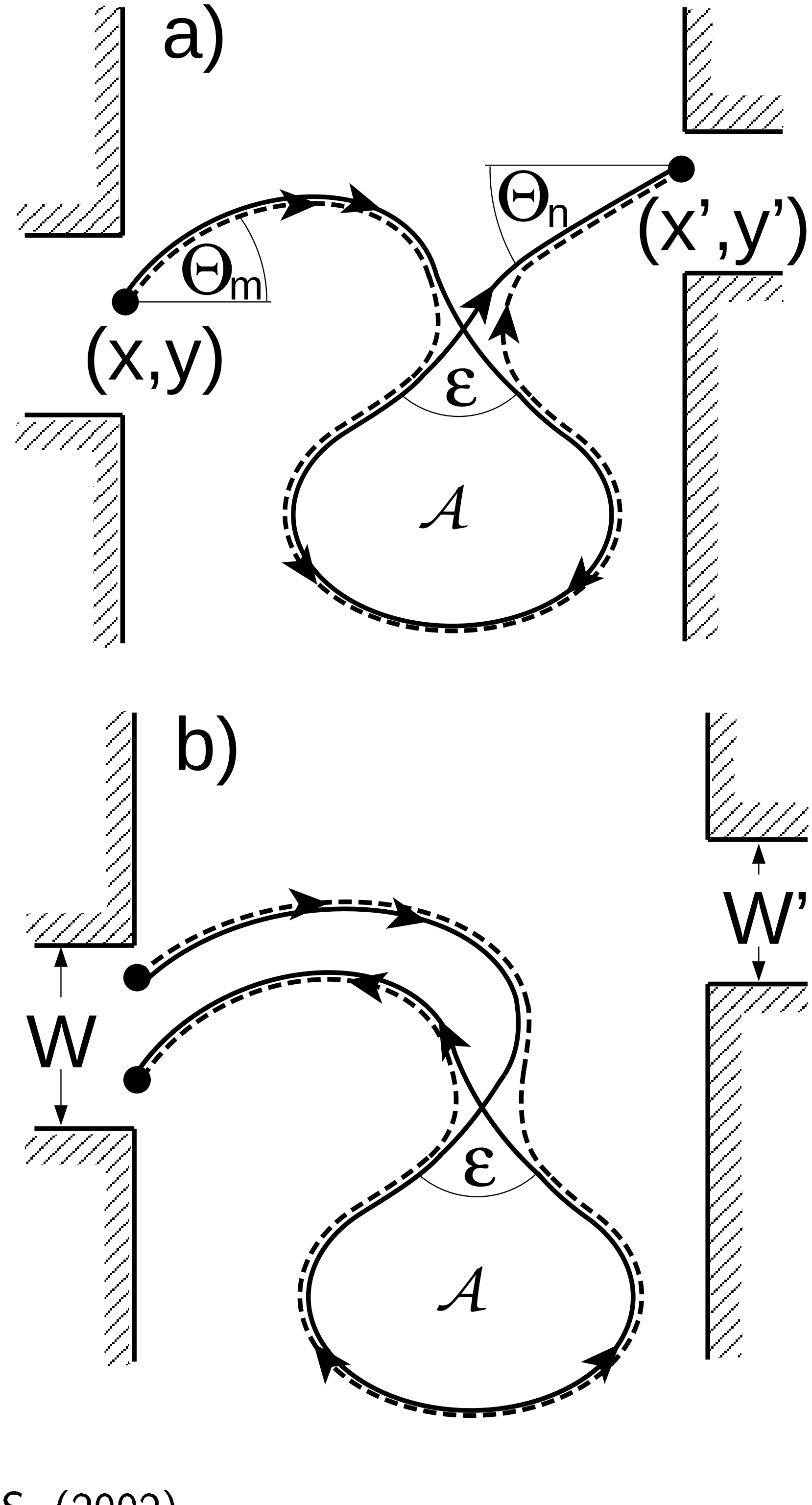
M.S., Richter (2001), M.S. (2002)

# Numerical search for loops



# Transport problem




Richter, M.S. (2002)

# Further work

- Quantum graphs (Berkolaiko, Schanz, Whitney, Gnutzmann, Altland)
- Nonuniformly hyperbolic systems (Turek, Richter, Spehner, Müller)
- Higher orders (Braun, Heusler, Müller, Haake, Altland)
- Other universality classes (Heusler, Nagao, Saito, Bolte, Harrison)
- Shot noise (Schanz, Puhlmann, Geisel)
- Open quantum systems (Puhlmann, Schanz, Kottos, Geisel)