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Intrinsic dissipative fluctuation rate in mesoscopic superconducting rings

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The rate at which dissipative fluctuations occur in narrow superconducting rings is considered within the Little-Langer-Ambegaokar-McCumber-Halperin framework. The mesoscopic regime, in which the ring circumference L does not greatly exceed the correlation length ξ, is specifically addressed. In this regime, significant differences arise between models that mimic voltage and current sources, the former (latter) exhibiting corrections to the fluctuation rate algebraic in ξ/L (exponentially small in L/ξ). The theory employs Forman's elaboration of the Gel'fand-Yaglom technique for computing fluctuation determinants.

Recent experiments have begun to investigate electrical resistance in narrow superconducting wires with lengths not greatly exceeding the temperature-dependent correlation length $\xi(T)$.[1] Motivated by these experiments, we address one particular aspect of this subject, namely the rate at which intrinsic dissipative (i.e., resistive) fluctuations of the order parameter occur in narrow superconducting rings. We focus, in particular, on the dependence of the rate on the circumference of the ring.

In the phase-slip picture of intrinsic resistive fluctuations in a narrow superconducting wire of length L, due to Little,[2] Langer and Ambegaokar,[3] and McCumber and Halperin,[4] which we refer to as the LAMH picture, the fluctuation rate $(L/\xi)\Gamma$ (Ref. 5) is constructed in terms of two physical quantities, a free-energy barrier height U and a (so-called) attempt frequency Ω, such that $(L/\xi)\Gamma = \Omega \exp\left(-U/k_BT\right)$. For wires of essentially infinite length, i.e., wires much longer than ξ, the fluctuation rate has been computed[3,4,6] and used to evaluate the phase-slip-induced resistance.

In this paper we extend the LAMH picture to the case of mesoscopic superconducting wires, i.e., wires for which the ratio ξ/L is not negligibly small. To this end, we analyze the length dependence of the barrier height and the attempt frequency, and hence the resistive fluctuation rate, for the case of a narrow superconducting closed ring of circumference L. The ring topology is adopted so as to preclude extrinsic effects associated with contacts to external circuitry. We envisage the sample as being threaded by a time-dependent magnetic flux $\hbar\Phi(t)/2e$ that induces an electromotive force $V(t) \equiv (\hbar/2e)\, d\Phi(t)/dt$ around the ring.

Two distinct situations are considered: either the system is driven by a voltage source or it is driven by a current source.[7] In the former case, $\Phi(t)$ is prescribed to increase linearly with time, i.e., $\Phi(t) = 2eVt/\hbar$, where V is the (constant) electromotive force. In the latter case, we envisage that the magnetic flux responds to the resistive fluctuations, by producing an electromotive force, in just such a manner so as to maintain a certain prescribed current.

The principal results of this paper concern the transition rates (i.e., the rates at which resistive fluctuations occur between current-carrying metastable states), and are twofold. First, for the case of a voltage source, we find that both the barrier heights and the attempt frequencies acquire corrections, algebraic in ξ/L, that increase them beyond their infinite L values. Thus, the corrections to the transition rates compete—an enhancement arising from the attempt frequencies and a suppression due to the barrier heights—the dominant correction being due to the barrier heights. Second, for the case of the current source, the transition rates are independent of the sample length (up to terms exponentially small in L/ξ). Although we focus here on the transition rates we note that the current-voltage characteristic can be constructed from them; see, e.g., Sec. 5 of Ref. 4.

Before embarking upon the analysis of the fluctuation rates, we remark that while the calculation of the barrier heights for a ring of arbitrary circumference is straightforward, the corresponding calculation of the attempt frequencies is, *prima facie*, quite challenging.

This task, however, is greatly facilitated, in fact becoming algorithmic in nature, by the use of Forman's recent generalization[8] of the technique of Gel'fand and Yaglom[9,10] for computing determinants of certain linear differential operators. It is a subsidiary aim of this paper to provide some exposure of this advance in technique, which we anticipate will find wide application.

I. VOLTAGE SOURCE

Following LAMH we consider a quasi-one-dimensional superconducting ring of cross-sectional area σ, circumference $L \equiv \ell\xi$ and (dimensionless) superconducting order parameter ψ. The Ginzburg-Landau free-energy functional F is given by $F[\psi] = gk_BT\,\mathcal{F}[\psi]$, where

$$\mathcal{F}[\psi] \equiv \int_{-\ell/2}^{\ell/2} dx \Big\{ |d\psi(x)/dx|^2 - |\psi(x)|^2 + \tfrac{1}{2}|\psi(x)|^4 \Big\}, \quad (1)$$

x is the longitudinal coordinate (measured in units of ξ) and $g \equiv \sigma H_c(T)^2 \xi(T)/4\pi k_B T$ (i.e., the condensation energy in a correlation length, measured in units of k_BT), in which $H_c(T)$ is the bulk critical field.[11,12] In the presence of a magnetic flux $\hbar\Phi/2e$ the boundary conditions are of the twisted periodic type, i.e.,

$$\psi(\ell/2) = e^{i\Phi}\psi(-\ell/2), \qquad \dot\psi(\ell/2) = e^{i\Phi}\dot\psi(-\ell/2), \quad (2)$$

where overdots denote spatial derivatives. Although Φ is not constant, we shall, following Ref. 3, assume that it varies on a time scale that is slow compared with the time scale of the resistive fluctuations.

Introducing the polar decomposition $\psi(x) = f(x)\,e^{i\phi(x)}$, the condition that $\mathcal{F}$ be stationary becomes

$$\dot E \equiv \partial_x\big[\dot f^2 + f^2 - \tfrac{1}{2}f^4 + f^2\dot\phi^2\big] = 0, \quad (3a)$$

$$\dot J \equiv \partial_x\big[f^2\dot\phi\big] = 0. \quad (3b)$$

The conserved quantity J is the dimensionless supercurrent, and is related to the physical supercurrent I by $I = 4Je(gk_BT/\hbar)$.

For the computation of the fluctuation rates there are two classes of important stationary states. First, there are the locally stable (i.e., metastable) states, given by

$$\psi_m(x;k_m,\phi_{m,0}) = f_m \exp i\phi_m(x), \quad (4a)$$

$$f_m^2 = u(k_m), \quad (4b)$$

$$\phi_m(x) = \phi_{m,0} + k_m x, \quad (4c)$$

where $u(q) \equiv (1-q^2)$, and $\phi_{m,0}$ is an arbitrary phase (which can be taken to be zero). These states are characterized by the discrete set of allowed wave vectors k_m satisfying $k_m\ell = 2\pi n_m + \Phi$, where n_m is integral, in terms of which the current J_m is given by $J_m = k_m\,u(k_m)$. Second, there are the transition (i.e., saddle-point) states, given by

$$\psi_s(x;k_s,m,x_0,\phi_{s,0}) = f_s(x)\exp i\phi_s(x), \quad (5a)$$

$$f_s(x)^2 = 2k_s^2 + \tfrac{1}{3}m_1\Delta(k_s) + m\,\Delta(k_s)\,\mathrm{sn}\big(\sqrt{\Delta(k_s)/2}\,(x-x_0)\big|m\big)^2, \quad (5b)$$

$$\phi_s(x) = \phi_{s,0} + J_s(k_s,m)\int_{x_0}^{x} dx' f_s(x')^{-2}, \quad (5c)$$

where $\phi_{s,0}$ is a second arbitrary phase and x_0 is an arbitrary position (which can both be taken to be zero), $\Delta(q) \equiv (1-3q^2)$ and $m_1 \equiv (1-m)$, and the current J_s is given by

$$2J_s(k_s,m)^2 = \big[u(k_s) + \tfrac{1}{3}m_1\Delta(k_s)\big]\big[u(k_s) - \tfrac{2}{3}m_1\Delta(k_s)\big] \times\big[2k_s^2 + \tfrac{1}{3}m_1\Delta(k_s)\big]. \quad (6)$$

The function sn in Eq. (5b) is a Jacobi elliptic function,[13,14] so that the boundary conditions, Eqs. (2), require $\ell(k_s,m) = \sqrt{8/\Delta(k_s)}\,K(m)$, in which K is the complete elliptic integral of the first kind.[14] These states are characterized by the allowed wave vectors k_s, which, up to terms of order m_1 (i.e., terms exponentially small in $\sqrt{\Delta/2}\,\ell$), satisfy $k_s\ell + 2\chi(k_s) = 2\pi n_s + \Phi$, where n_s is integral and $\chi(q) \equiv \arctan\sqrt{\Delta(q)/2q^2}$.

Imposing the requirement that the current-decreasing (-increasing) transition from a metastable state k_m occur via the transition state k_s^- (k_s^+) with the same (2π larger) total winding angle $\Delta\phi \equiv \int_{-\ell/2}^{\ell/2} dx\,\dot\phi$ as the initial metastable state, one identifies that the appropriate transition state $k_s^\pm$ satisfies $k_m\ell = k_s^\pm\ell + 2\chi(k_s^\pm) - (\pi\pm\pi)$, up to terms of order m_1.

Having identified the transition state k_s^- associated with current-reducing fluctuations out of the metastable state k_m, it is straightforward to establish that the barrier height U for current-reducing fluctuations[15] out of k_m is given by $U = gk_BT\big(\mathcal{F}[\psi_s] - \mathcal{F}[\psi_m]\big) \equiv gk_BT\Delta\mathcal{F}$, where[16]

$$\Delta\mathcal{F} = \Delta\mathcal{F}^{(0)} + \ell^{-1}\Delta\mathcal{F}^{(1)} + \mathcal{O}(\ell^{-2}), \quad (7a)$$

$$\Delta\mathcal{F}^{(0)} = \tfrac{4}{3}\sqrt{2\Delta(k_m)} - 4\chi(k_m)k_m u(k_m), \quad (7b)$$

$$\Delta\mathcal{F}^{(1)} = 4\chi(k_m)^2\Delta(k_m). \quad (7c)$$

Although we exhibit only the $\mathcal{O}(\ell^{-1})$ correction to the infinite sample-length value, higher-order corrections can readily be computed. Evidently, the barrier heights are increased relative to their infinite-length value. That the correction is algebraic in ℓ^{-1}, rather than exponential, results from the presence of the phase degree of freedom.

We now turn to the calculation of the attempt frequency Ω,[17,18,4] given by

$$\Omega = \omega(T)\,\frac{\mathcal{V}_s^{(1)}\mathcal{V}_s^{(2)}}{\mathcal{V}_m^{(1)}}\,|\lambda_{s0}|\left|\frac{\det' L_s}{\det' L_m}\right|^{-1/2}, \quad (8)$$

where $\omega(T) \equiv \sqrt{g/4\pi^3}/\tau(T)$ is a characteristic frequency, in which $\tau(T)$ is the temperature-dependent relaxation rate. The $\mathcal{V}$ factors arise from integrations over the so-called zero modes[19] (resulting from translational and gauge invariance), λ_{s0} is the (negative) eigenvalue (associated with the reaction coordinate), and the subscripts m and s, respectively, denote the metastable and saddle-point states. The quotient $\det' L_m/\det' L_s$ is the quotient of determinants (with zero eigenvalues omitted) of the operators L_e (with e being m or s) that are associated with the second variation of the free energy, $\delta\mathcal{F}_e^{(2)} = \int_{-\ell/2}^{\ell/2} dx\,\delta\Psi^\dagger L_e\,\delta\Psi$. The two components of $\delta\Psi$

are the order parameter fluctuation $\delta\psi$ and its complex conjugate, so that

$$L_e = \begin{bmatrix} -\partial_x^2 - 1 + 2f_e^2 & f_e^2\, e^{2i\phi_e} \\ f_e^2\, e^{-2i\phi_e} & -\partial_x^2 - 1 + 2f_e^2 \end{bmatrix}. \tag{9}$$

In order to calculate $\det' L_s/\det' L_m$ we make use of a generalization, due to Forman,[8] of the Gel'fand-Yaglom technique, suitable for 2×2-matrix differential operators:

$$\frac{\det L_s}{\det L_m} = \frac{\det\left[M + NY_s(\ell/2)\right]}{\det\left[M + NY_m(\ell/2)\right]}. \tag{10}$$

Here, the (4×4) matrix $Y_e(x)$ is the so-called fundamental matrix, whose construction is described below, and the (4×4) matrices N and M encode the boundary conditions.[20] Thus, the quotient of the determinants of two infinite-dimensional matrices is reduced to the quotient of the determinants of two finite-dimensional matrices. For the case of twisted periodic boundary conditions, Eqs. (2), considered here, N is the (4×4) identity matrix, and $-M$ is the (4×4) diagonal matrix, $\mathrm{diag}\{e^{i\Phi}, e^{-i\Phi}, e^{i\Phi}, e^{-i\Phi}\}$. Not unusually, the quotient $\det L_s/\det L_m$ is ill-defined[19] due to the presence of unwanted zero modes, which we eliminate using the following strategy. First, we regularize the operators, i.e., perturb them so as to eliminate any zero eigenvalues, by adjusting the continuous parameters k_e. Second, we compute the quotient of regularized determinants, using Eq. (10). Third, we compute perturbatively the formerly zero eigenvalues. Fourth, we factor out the formerly zero eigenvalues from the regularized quotient of determinants. Fifth, we remove the regularization. Thus we obtain $\det' L_s/\det' L_m$.

At the heart of the strategy outlined in the previous paragraph is the computation of the fundamental matrix $Y_e(x)$, which is defined to have the property that, for any (complex two-component) solution $\zeta_e(x)$ of the so-called Jacobi accessory equation $L_e\,\zeta_e = 0$, one has

$$\begin{bmatrix} \zeta_e(x) \\ \partial_x\zeta_e(x) \end{bmatrix} = Y_e(x) \begin{bmatrix} \zeta_e(-\ell/2) \\ \partial_x\zeta_e(-\ell/2) \end{bmatrix}. \tag{11}$$

We have found it convenient to express $Y_e(x)$ in terms of an auxiliary (4×4) matrix $H_e(x)$, via $Y_{e,ij}(x) = \sum_{k=1}^{4} H_{e,ik}(x)H^{-1}_{e,kj}(-\ell/2)$, where the ith column of $H_e(x)$ is the (four-component) vector $[\eta_{e,i}(x), \dot\eta_{e,i}(x)]$, in which $\{\eta_{e,i}\}_{i=1}^4$ are four linearly independent solutions of the accessory equation. It is elementary to obtain the solutions of the accessory equation $\{\eta_{m,i}\}$ at the metastable states, but not the solutions $\{\eta_{s,i}\}$ at the saddle-point states. However, the latter may be constructed algorithmically, using Jacobi's theorem,[21] i.e., by differentiating the general solution to the Euler-Lagrange stationarity condition, ψ_s of Eq. (5a), with respect to each of the four constants of integration $\{k_s, m, x_0, \phi_{s,0}\}$. Using this representation of $Y_e(x)$, together with the symmetry properties of the elements of $H_e(x)$, we readily obtain $Y_e(\ell/2)$. Elimination of zero modes, as outlined above, then yields $\det' L_s/\det' L_m$. We wish to stress the algorithmic nature of this approach, which requires solely the general solution of the Euler-Lagrange stationarity condition.

Upon following the strategy described above for computing the attempt frequency for current-reducing (-increasing) fluctuations $\Omega^{-(+)}$ of Eq. (8) becomes[22]

$$\Omega^{\pm} = 2^{7/4}\omega(T)\ell\Lambda(k_s^{\pm})\Delta(k_s^{\pm})^{7/4}\, e^{\sqrt{\Delta(k_m)/2}\ell - \sqrt{\Delta(k_s^{\pm})/2}\ell} \times \left[\frac{\Delta(k_s^{\pm})}{\Delta(k_m)}\frac{u(k_m)}{u(k_s^{\pm})}\right]^{1/2} \left[1 - \frac{2\sqrt{2}}{u(k_s^{\pm})\sqrt{\Delta(k_s^{\pm})}}\frac{1}{\ell}\right]^{-1/2}, \tag{12}$$

where $\Lambda(q) \equiv -1-q^2+\left[3\Delta(q)^2+(1+q^2)^2\right]^{1/2}$. A straightforward examination of this function indicates that the attempt frequency per unit length increases as ℓ is reduced, behaving algebraically in ℓ^{-1}. To probe experimentally this sensitivity to length, it may be useful to consider current-relaxation experiments at fixed flux. For example, the ring may be prepared in a current-carrying state, with the rate at which the current decays being measured inductively.

II. CURRENT SOURCE

To describe the case of a current source, we no longer restrict the ensemble of superconducting states to those satisfying Eqs. (2) with Φ fixed. Instead, we consider the unrestricted ensemble of states satisfying Eqs. (2) *but with the flux Φ free to vary.* As the system is not in the thermodynamic limit, it is necessary to generalize the strategy employed by McCumber,[7] accounting for the enlarged ensemble by introducing an appropriate fugacity and integrating over the (now unconstrained) variable. Hence, the formula for the transition rate[18] becomes

$$(L/\xi)\Gamma = \omega(T)\frac{\displaystyle\int_0^{2\pi} d\Phi\, e^{2gJ\Phi}\left\{e^{-g\mathcal{F}[\psi_s]}\,\mathcal{V}_s^{(1)}\mathcal{V}_s^{(2)}|\lambda_{s0}|\left|\det' L_s\right|^{-1/2}\right\}\Big|_{\Phi}}{\displaystyle\int_0^{2\pi} d\Phi\, e^{2gJ\Phi}\left\{e^{-g\mathcal{F}[\psi_m]}\,\mathcal{V}_m^{(1)}\left|\det' L_m\right|^{-1/2}\right\}\Big|_{\Phi}}, \tag{13}$$

where the (generalized) fugacity $\exp(2gJ)$ controls the mean winding angle.

In the calculation of the rate, using Eq. (13), the Φ integrations may be performed using Laplace's method because $g \gg 1$.[11] The maximum values of the exponents, i.e., (minus) the Gibbs free energies[7] $\mathcal{G}[\psi_e] \equiv \mathcal{F}[\psi_e] - 2J\Phi_e$, occur at $\Phi_s = k\ell + 2\chi(k) \pmod{2\pi}$ and $\Phi_m = k\ell \pmod{2\pi}$, where $k(< 1/\sqrt{3})$ satisfies $J = k\,u(k)$. In other words, the current source responds to a fluctuation by inserting (or removing) flux so as to maintain a constant current. Thus, Φ_m and Φ_s differ, in contrast with the case of the voltage source. The computation of the appropriate fluctuation determinants proceeds along precisely the same lines as for the voltage-source case, although Φ_m and Φ_s now differ. For flux-increasing (i.e., positive electromotive force) fluctuations[7,23] we obtain

$$(L/\xi)\Gamma = 2^{7/4}\omega(T)\ell\Lambda(k)\Delta(k)^{7/4}e^{-U/k_BT}, \qquad (14)$$

where the barrier heights U are given by $U = gk_BT(\mathcal{G}[\psi_s] - \mathcal{G}[\psi_m]) \equiv gk_BT\Delta\mathcal{G}$, where

$$\Delta\mathcal{G} = \tfrac{4}{3}\sqrt{2\Delta(k)} - 4\chi(k)k\,u(k) + \mathcal{O}(m_1). \qquad (15)$$

Although the attempt frequency factors in the integrands of Eq. (13) depend algebraically on ξ/L [cf. Eq. (12)], this algebraic dependence is canceled by terms arising from Gaussian fluctuations of Φ. Thus, in contrast with the voltage-source case, we see that the rate per unit length Γ is insensitive to the sample circumference, up to terms exponentially small in L/ξ.

III. CONCLUSIONS

We have addressed the length dependence of the dissipative transition rate $(L/\xi)\Gamma$ for a mesoscopic superconducting ring. For the voltage-source case, we have found that the leading length-dependent corrections to the barrier heights and attempt frequencies are algebraic in ξ/L. To elucidate the implications of these length dependences for current-decreasing fluctuations it is useful to expand, viz., $\Delta\mathcal{F} \approx \Delta\mathcal{F}^{(0)} + \ell^{-1}\Delta\mathcal{F}^{(1)}$ and $\Omega \approx \Omega^{(0)} + \ell^{-1}\Omega^{(1)}$. Thus, we find $\ln[\Gamma(\ell)/\Gamma(\infty)] \approx \ell^{-1}\{-g\Delta\mathcal{F}^{(1)} + \Omega^{(1)}/\Omega^{(0)}\}$, where $\Gamma(\infty) \equiv \Omega^{(0)}\exp(-g\Delta\mathcal{F}^{(0)})$. As both $\Delta\mathcal{F}^{(1)}$ and $\Omega^{(1)}$ are positive, a competition between them arises. However, for the LAMH picture to be accurate one should have $g \gg 1$. Thus, for the case of the voltage source the length dependence of the transition rate is essentially determined by the length dependence of the barrier height, resulting in a *decrease* in the rate per unit length as ℓ is reduced. For the sake of illustration, we suppose that $T_c = 1\,\mathrm{K}$, $H_c = 100\,\mathrm{G}$, $\xi(0) = 1000$ Å, $\sqrt{\sigma} = 750$ Å, and $T/T_c = 0.99$, so that $g \approx 9.8$. In the low-current limit, $\Delta\mathcal{F}^{(1)} = \pi^2$, so that $\Gamma(\ell)/\Gamma(\infty) \approx \exp(-g\Delta\mathcal{F}^{(1)}/\ell) \approx \exp(-99/\ell)$, i.e., roughly 0.14 when $L = 50\,\mu$m. In contrast, for the current-source case, we observe that the transition rate per unit length Γ is essentially *independent* of L (i.e., independent of L up to terms exponentially small in L/ξ). We therefore encounter significant differences between two distinct experimental configurations—constant voltage versus constant current—as a result of the mesoscopic size of the system.

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[23]To obtain the barrier height for flux-decreasing fluctuations one must replace χ by $\chi - \pi$.