Extinction of metastable stochastic populations

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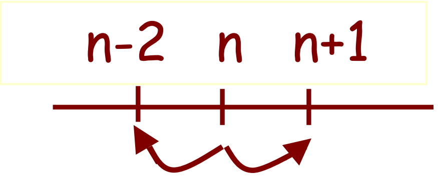
Outline

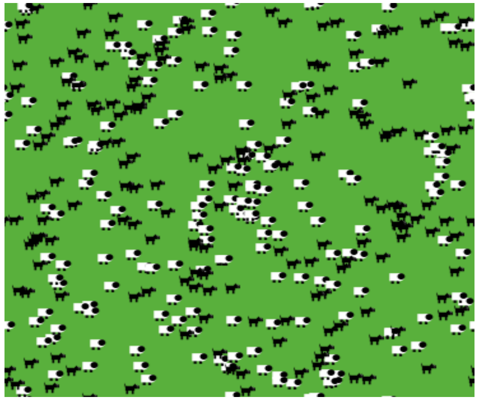
- ✓ Stochastic population dynamics: introduction
- ✓ Extinction of a self-regulating population in zero dimensions
- ✓ WKB approximation

+ more briefly:

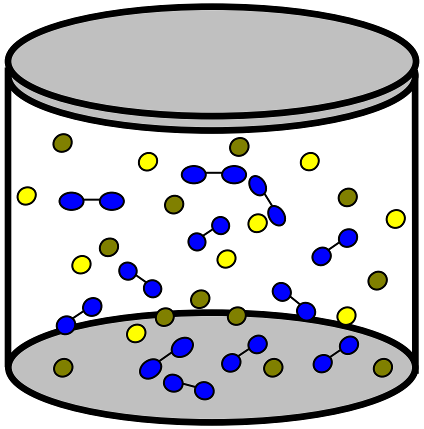
- ✓ Multiple species: extinction of an epidemic
- ✓ Population extinction in a fluctuating environment

Stochastic population dynamics = Markov process on discrete set of states: 0,1,2, …





stochastic models of
population biology,
ecology, epidemiology



biochemistry:
intracellular processes

astrochemistry:
reactions on dust grains
in interstellar medium

many more examples

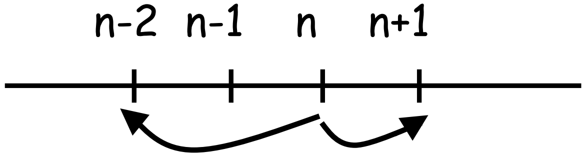
Toy model: **branching-annihilation reaction**

$$A \xrightarrow{\lambda} 2A$$

$$2A \xrightarrow{\mu} 0$$

isolated population

Turner and Malek-Mansour 1978, Kessler and Shnerb 2007, Assaf and M 2007



single-population
Markov process

$P_n(t)$: probability to find n particles A at time t

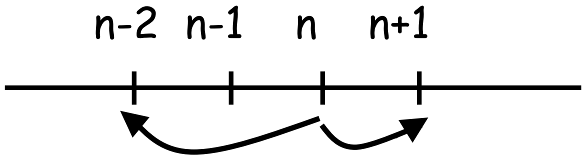
$P_0(t)$: probability of extinction at time t

$$\begin{cases} \dfrac{dP_n}{dt} = \underbrace{\dfrac{\mu}{2}\left[(n+2)(n+1)P_{n+2} - n(n-1)P_n\right]}_{\text{annihilation}} + \underbrace{\lambda\left[(n-1)P_{n-1} - nP_n\right]}_{\text{branching}}, \quad n = 1,2,3,\dots \\ \dfrac{dP_0}{dt} = \mu P_2 \end{cases}$$

Go back to **branching-annihilation** model

$$A \overset{\lambda}{\rightarrow} 2A$$

$$2A \overset{\mu}{\rightarrow} 0$$



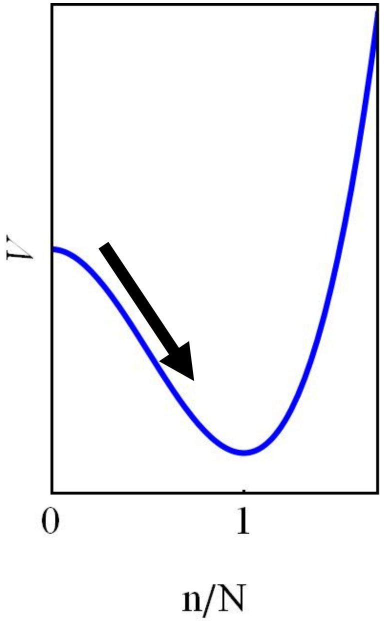
What does the *rate equation* predict?

$$\dot{\bar{n}}(t) \cong \lambda \bar{n}(t) - \mu \bar{n}^2(t) \Longrightarrow$$

repelling fixed point $n_* = 0$

attracting fixed point $n_s = \frac{\lambda}{\mu} \equiv N >> 1$

large parameter



over-damped "particle"
in potential

$$V(n) = -\frac{\lambda}{2} n^2 + \frac{\mu}{3} n^3$$

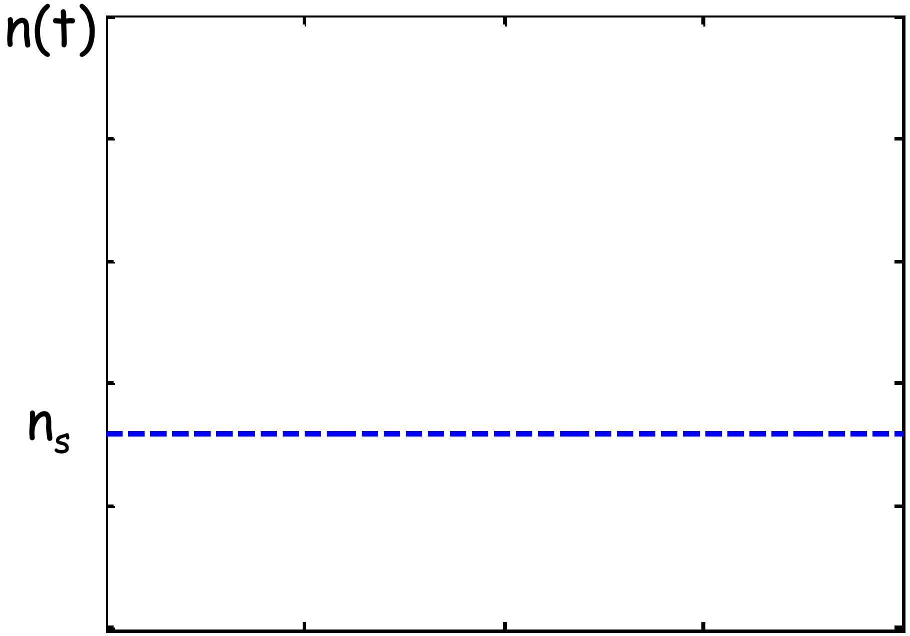
Population size flows to n_s=N and stays there forever…

This prediction breaks down at long times

$$A \xrightarrow{\lambda} 2A$$

$$2A \xrightarrow{\mu} 0$$

Monte Carlo simulation



a fatal large fluctuation brings population to absorbing state n=0: *extinction*

Interesting to find:

- mean time to extinction (MTE)
- distribution of extinction times
- *quasi*-stationary probability distribution (QSD) of population sizes

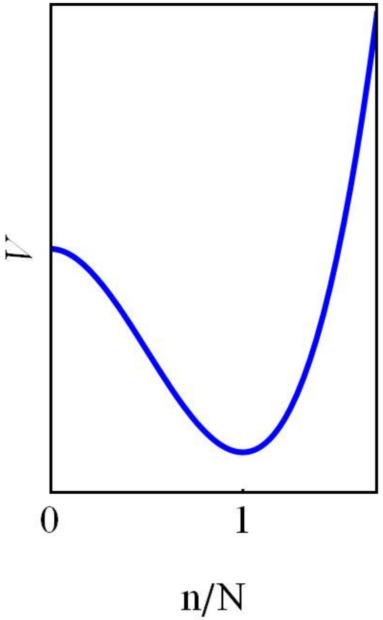
Rate equation enables one to **classify** extinction scenarios of metastable populations

$$\dot{\bar{n}} = \bar{n} \cdot \Phi(\bar{n}) \qquad \bar{n} = 0 \begin{cases} \text{repelling fixed point: } \textbf{scenario A} \\ \text{attracting fixed point: } \textbf{scenario B} \end{cases}$$

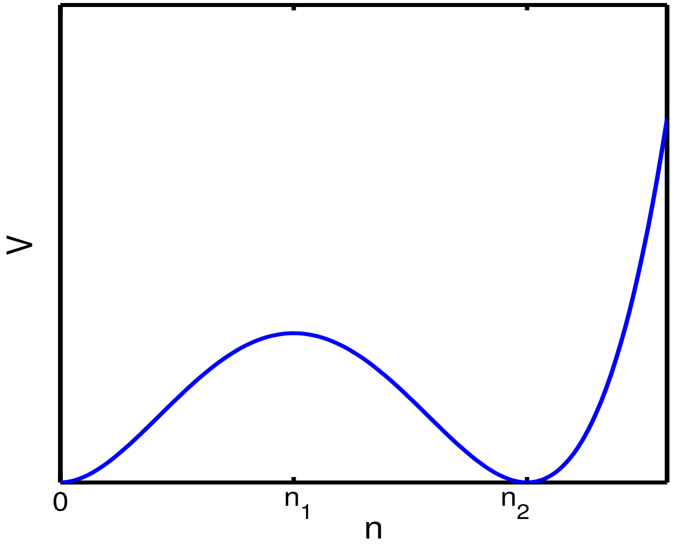
Over-damped "particle" in a potential

$$\dot{\bar{n}} = -\frac{d}{dn} V(\bar{n})$$

Scenario A



Scenario B



Crossing entropy barrier takes exponentially long time

The information we seek is encoded in the master equation

$$\frac{dP_n(t)}{dt} = \sum_r W_r(n-r)P_{n-r}(t) - W_r(n)P_n(t)$$

W - transition rate matrix

How to solve the master equation analytically?

Widely used tool: approximating the master equation by Fokker-Planck equation

Fokker-Planck equation: example

$$A \xrightarrow{\lambda} 2A$$

$$2A \xrightarrow{\mu} 0$$

$$\frac{dP_n}{dt} = \frac{\mu}{2}\left[(n+2)(n+1)P_{n+2} - n(n-1)P_n\right] + \lambda\left[(n-1)P_{n-1} - nP_n\right]$$

Assume n>>1 and expand in Taylor series up to the second order in 1/N

$$\frac{\partial P(n,t)}{\partial t} \cong \frac{\mu}{2}\frac{\partial}{\partial n}\left\{-2n(N-n)P(n,t) + \frac{1}{2}\frac{\partial}{\partial n}\left[2n\left(2n+N\right)P(n,t)\right]\right\}$$

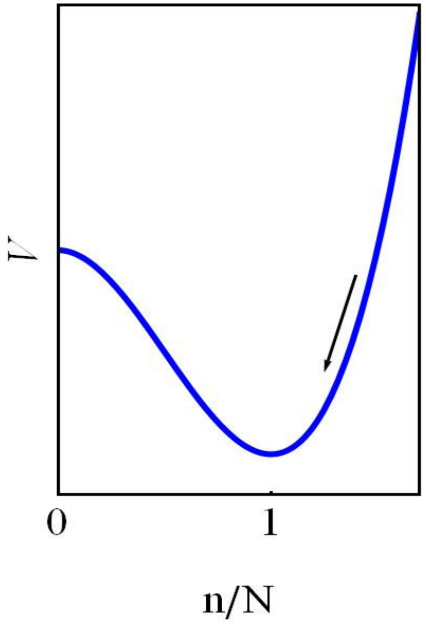
drift

diffusion

Drift term corresponds to deterministic rate equation

$$\dot{n} = \mu n(N-n)$$

over-damped "particle" in potential V



Diffusion broadens the distribution around n=N and drives the effective "particle" toward n=0, overcoming *entropic barrier to extinction*

The FP-approximation looks reasonable…

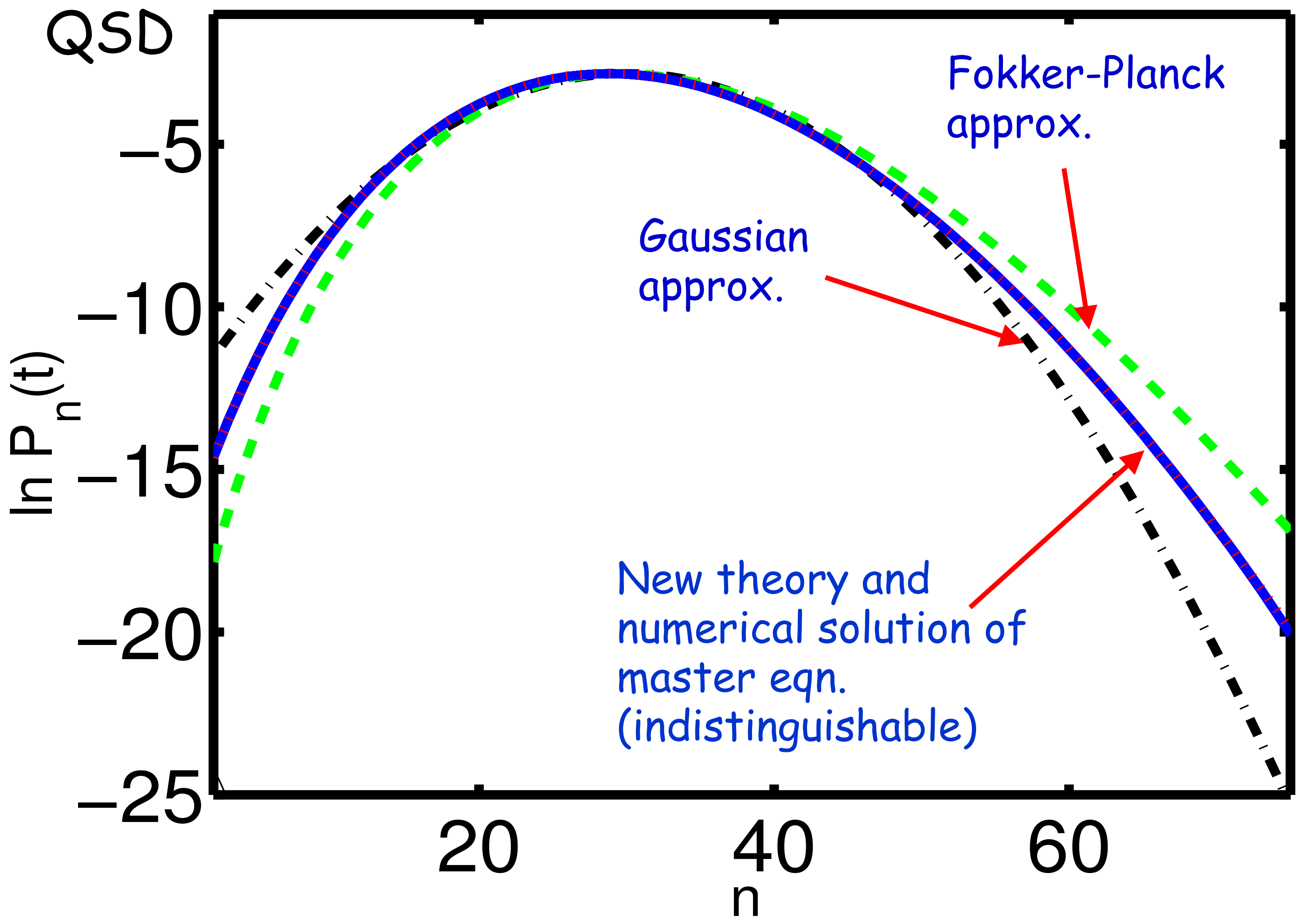
However, it is wrong: the error in the mean time to extinction, which it predicts, is exponentially large in N.

…, Gaveau, Moreau and Toth 1996, Doering, Sargsyan and Sander 2005, Kessler and Shnerb 2007, Assaf and M 2007, …

The error comes from a wrong description of distribution *tails*

$$A \xrightarrow{\lambda} 2A$$

$$2A \xrightarrow{\mu} 0$$



Fokker-Planck approx.

Gaussian approx.

New theory and numerical solution of master eqn. (indistinguishable)

Assaf and M 2007, Kessler and Shnerb 2007

Eigenvalue problem for master equation

Spectral expansion $$P_n(t) = \delta_{0n} + \sum_{k=1}^{\infty} \pi_n^{(k)} e^{-E_k t}$$

Metastability: two widely different time scales $t_r << \tau$

relaxation time – decay of higher eigenmodes

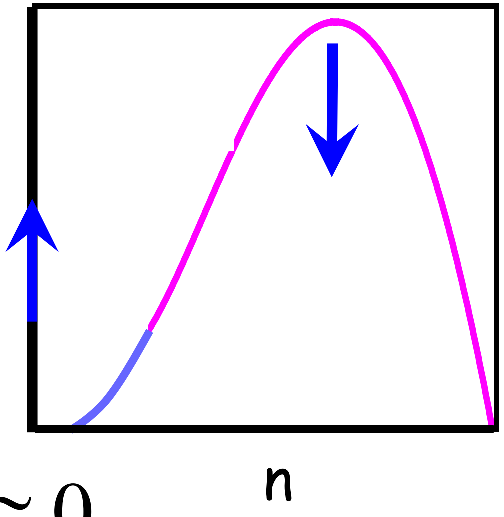
extinction time – decay of metastable state = first excited eigenmode

For t>>t_r

$$P_0(t) \approx 1 - e^{-t/\tau}$$

$$P_{n>0}(t) \approx \pi_n e^{-t/\tau}$$

quasi-stationary distribution π_n



$$\Downarrow$$

$$\sum_r W_r(n-r)\pi_{n-r} - W_r(n)\pi_n \cong 0$$

quasi-stationary master equation

WKB ansatz

$$\pi(n) \equiv \pi(qN) = e^{-NS(q)-S_1(q)-O(1/N)}$$

$N >> 1$

$n >> 1$

$q = n/N$

Dykman *et. al.* 1995, Kessler and Shnerb 2007, M and Sasorov 2008, Escudero and Kamenev 2009, Assaf and M 2009

rescaling of time →

$$W_r(n) = Nw_r(q) + u_r(q) + O(1/N)$$

$$W_r(0) = 0$$ ***absorbing state***

Systematic perturbation theory order by order in 1/N

Leading-order WKB approximation for action S(q)

$$H(q,p)=0$$ where $$H(q,p)=\sum_r w_r(q)\left(e^{rp}-1\right)$$

$$p=S'(q)$$ momentum

Look for zero-energy phase trajectories of a Hamiltonian flow

zero-energy lines:

$$p=p_s=0$$

$$p=p_f(q)\neq 0$$

corresponding actions are

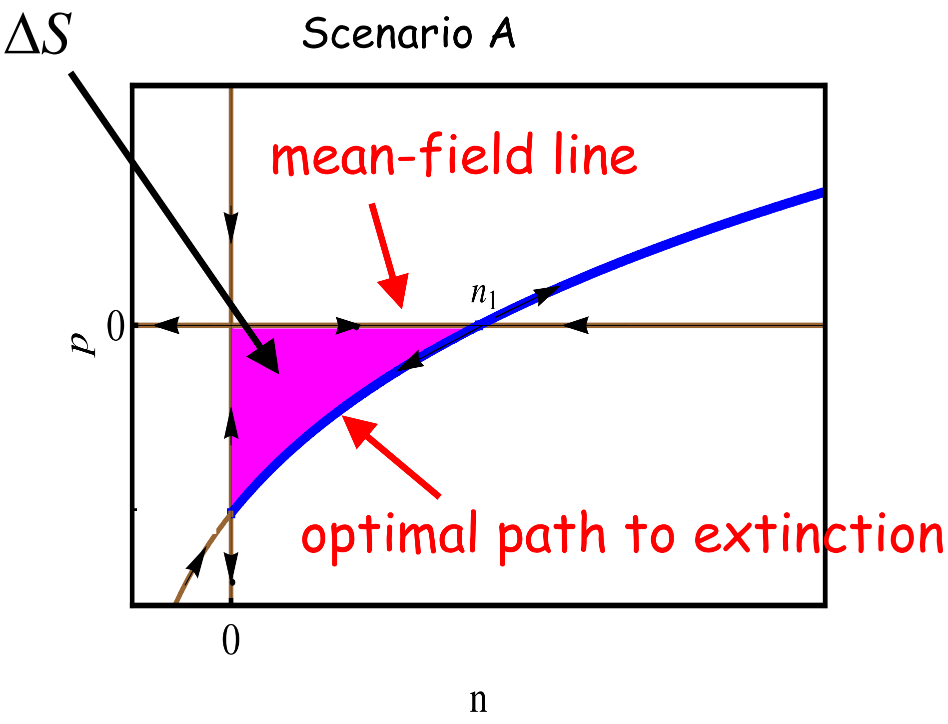
$$S^{(f)}(q)=\int^{q} p_f(q')dq'$$ fast-mode solution

$$S^{(s)}(q)=0$$ slow-mode solution

Escape occurs in extended phase plane (n,p)

$(n_{1,2}, p = 0)$ saddles

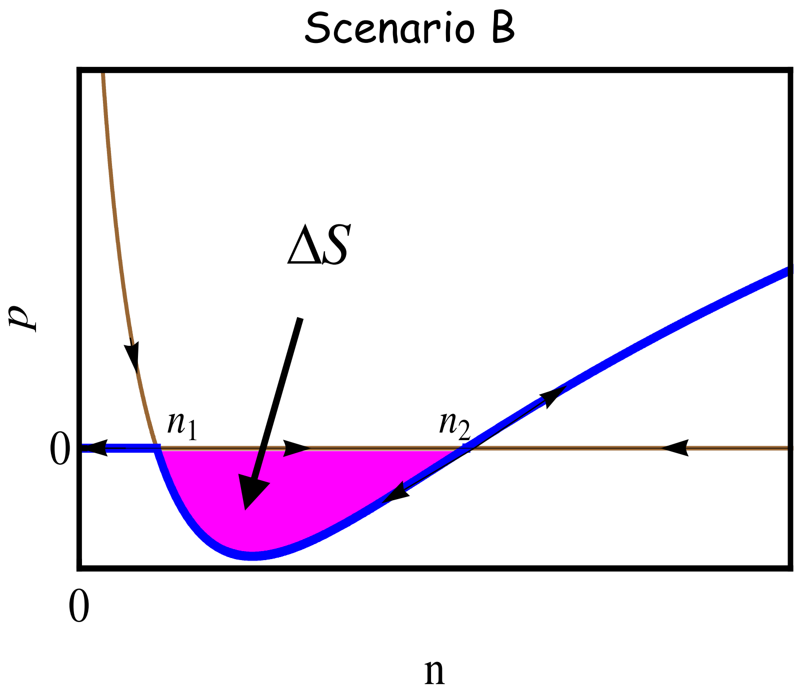
p - "fluctuational" coordinate



Scenario A

mean-field line

optimal path to extinction



Scenario B

In the leading order $\tau \sim e^{N\Delta S}$ Elgart and Kamenev (2004)

Sub-leading WKB order yields amplitude $S_1(q)$

Solving 1st-order ODE for $S_1(q)$:

$$S_1^{(f)}(q)=\int\frac{H_{pq}(q,p_a)+1/2\,H_{pp}(q,p_a)\,p_a'(q)-\sum_r u_r(q)(e^{rp_a}-1)}{H_p(q,p_a)}dq$$

$$S_1^{(s)}(q)=\int\frac{H_{pq}(q,0)}{H_p(q,0)}dq=\ln H_p(q,0)$$

WKB solution, valid for n>>1, is a linear combination of fast and slow modes

WKB breaks down at n=O(1). Here: recursion solution of quasi-stationary master equation

MTE and complete QSD can be found by matching WKB solution to recursion solution

Matching is scenario-dependent

Recursion solution at small n

At n<<n$_1$, linear terms in n suffice: $w_r(n) \approx n w_r\,'(0)$

$$\sum_r R_r\left[(n-r)\pi_{n-r} - n\pi_{n-r}\right] = 0 \qquad R_r \equiv w_r\,'(0)$$

linear branching rates

$$A \xrightarrow{R_r} (r+1)A$$

By putting $\pi_n = f_n/n$ the problem reduces to finding all roots of a polynomial equation

To find MTE, one only needs n>>1 asymptote of recursive solution, to be matched with WKB solution

For n>>1 $\pi_n \approx \frac{1}{\alpha\tau n}\begin{cases} C_1 e^{-np_f(0)} & \text{scenario A} \\ C_0 & \text{scenario B} \end{cases}$

$$C_0, C_1 = \Psi_{0,1}\left(R_{-1}, R_1, \ldots\right)$$

$$p_f(0) < 0$$

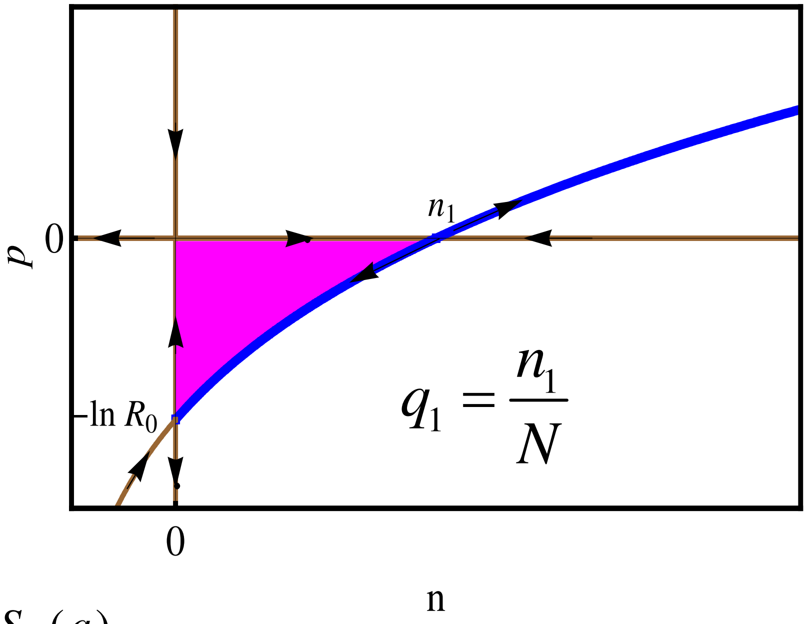
Matching the solutions: Scenario A

Here fast mode dominates in entire region of validity of WKB solution

Slow mode gives exponentially small correction and must be neglected

First we normalize QSD to unity

Close to maximum of distribution $n_1 \gg 1$,
WKB solution holds



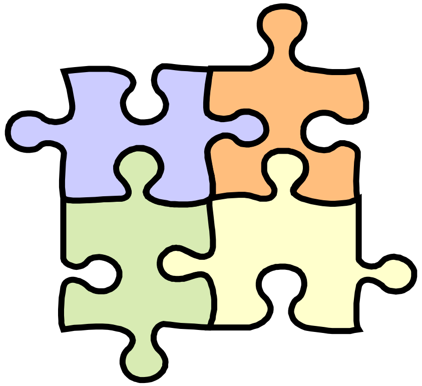
$$q_1 = \frac{n_1}{N}$$

$$\pi(q) = \sqrt{\frac{S''(q_1)}{2\pi N}}\, e^{N[S(q_1)-S(q)]+S_1(q_1)-S_1(q)}$$

Close to q=0

$$S(q \approx 0) = S(0) + qS'(0) = S(0) + qp_f(0)$$

$$\pi(n)^{(WKB)} \sim e^{N[S(q_1)-S(0)]-np_f(0)} \qquad 1 << n << N^{1/2}$$



$$\pi(n)^{(recursion)} \sim \frac{e^{-np_f(0)}}{\tau} \qquad 1 << n << N^{1/2}$$

$$\longrightarrow \qquad \tau \sim e^{N[S(0)-S(q_1)]} = e^{N\Delta S}$$

Result with pre-exponent

$$\boxed{\tau = \frac{A_1}{\alpha q_1}\sqrt{\frac{2\pi}{NS''(q_1)}}e^{N[S(0)-S(q_1)]+\Phi(0)-\Phi(q_1)}}$$

A_1 related to C_1

$\Phi(q)$ related to $S_1(q)$

Particular case: single-step processes

➡ the only nonzero reaction rates $n \xrightarrow{W_\pm(n)} n\pm 1$ $\qquad W_\pm(n) \approx N w_\pm(q) + u_\pm(q)$

Here the calculations are very simple

$$H(q,p) = w_+(q)(e^{p} - 1) + w_-(q)(e^{-p} - 1) \qquad p_f(q) = \ln \frac{w_-(q)}{w_+(q)}$$

$$\boxed{\tau = \frac{\sqrt{2\pi R_1}\; \mathrm{e}^{\int_0^{q_1} (u_+/w_+ - u_-/w_-)dq}}{\alpha(R_1 - 1)\sqrt{NS''(q_1)}} \quad e^{N\int_0^{q_1} \ln\left(\frac{w_+}{w_-}\right)dq}} \qquad A \xrightarrow{R_1} 2A$$

Example: SIS model of epidemics

$$S + I \xrightarrow{\lambda} 2I$$

$$I \xrightarrow{\mu} S$$

holds for short epidemic durations when renewals/removals can be neglected

$$I \xrightarrow{\lambda} 2I$$

$$I \xrightarrow{\mu} \varnothing$$

$$W_{+1}(n) = \lambda n(N-n)$$

$$W_{-1}(n) = \mu n$$

$$q_1 = n_1/N = 1 - 1/R_0$$

$$R_0 = \lambda N/\mu$$

$$\int_0^{q_1} \ln\left(\frac{w_+}{w_-}\right) dq = \ln R_0 + \frac{1}{R_0} - 1$$

$$\int_0^{q_1} \left(u_+/w_+ - u_-/w_-\right) dq = 0$$

$$\tau = \frac{1}{\mu}\sqrt{\frac{2\pi}{N}}\frac{R_0}{\left(R_0-1\right)^2}\exp\left[N\left(1-\frac{1}{R_0}-\ln R_0\right)\right]$$

coincides with Nasell (2001)

Scenario B

Slow mode is dominant on the interval (0,q_1). Finding complete WKB solution demands matching between fast- and slow-mode solutions

Example: chemical bistability

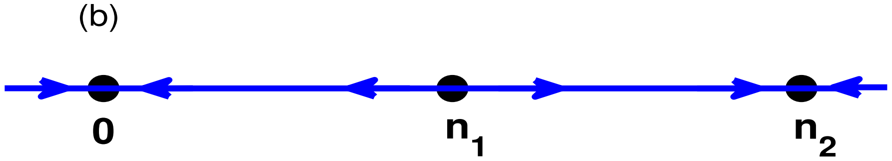
$$A \xrightarrow{\mu} \varnothing$$

$$2\,A \xrightarrow{\lambda} 3\,A$$

$$3\,A \xrightarrow{\sigma} 2\,A$$

Rate equation:

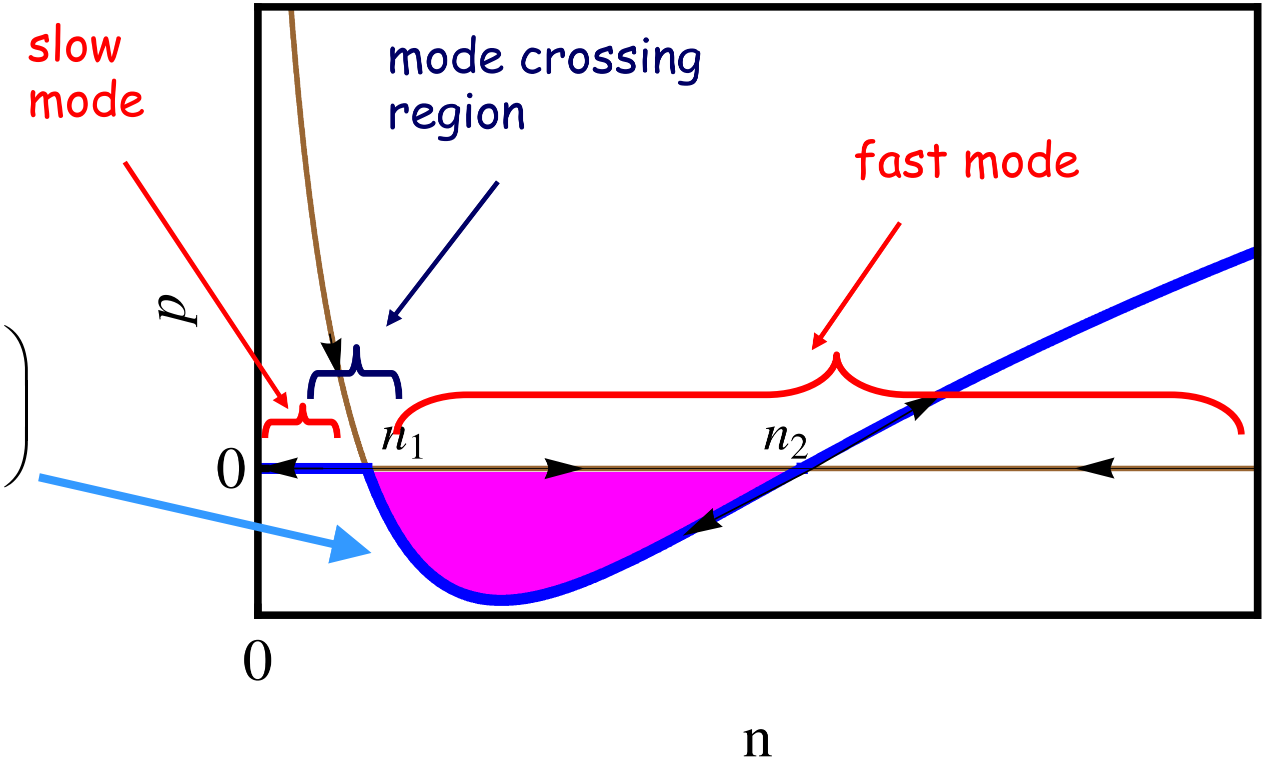
$$\dot{\bar{n}}(t) = -\mu\bar{n}(t) + \frac{\lambda}{2}\bar{n}^2(t) - \frac{\sigma}{6}\bar{n}^3(t)$$



$n=0$ $\quad n_1 = N(1-\delta)$ $\quad n_2 = N(1+\delta)$

$$N = \frac{3\lambda}{2\sigma}, \qquad \delta^2 = 1 - \frac{8\sigma\mu}{3\lambda^2}$$

$$p_f(q) = \ln\left(\frac{q}{2} + \frac{1-\delta^2}{2q}\right)$$



WKB solution includes three regions

$q > q_1, \quad q - q_1 >> N^{-1/2}$	fast mode (nonzero action) dominant
$q < q_1, \quad q_1 - q >> N^{-1/2}$	slow mode (zero action) dominant
$\lvert q - q_1 \rvert << 1$	mode-coupling region

How to match the fast and the slow modes? No joint region!

Solve quasi-stationary master equation in vicinity of $q \approx q_1$. Here p is small and so FP equation is valid

Employing van-Kampen system size expansion, for $n \gg 1$

$$\pi(n \pm 1) \approx \pi(n) \pm \pi'(n) + \frac{1}{2}\pi''(n)$$

Constant-current Fokker-Planck equation

FP solution at $N^{-1/2} << q_1 - q << 1$ matched with slow-mode solution

FP solution at $N^{-1/2} << q - q_1 << 1$ matched with fast-mode solution

↓

Complete WKB solution

Slow mode vital for finding pre-exponent

Finally, matching WKB solution with recursion solution at n=O(1)

in chemical
bistability example:

$$\pi_n = \frac{\pi_1}{n} = \frac{1}{\mu\tau n}$$

$$\tau = \frac{\delta}{\pi\mu(1-\delta)} e^{N\Delta S} \qquad \Delta S = 2\left(\delta - \sqrt{1-\delta^2}\,\arctan\frac{\delta}{\sqrt{1-\delta^2}}\right)$$

General result for scenario B:

$$\tau = \frac{4\pi}{\alpha H_{pp}(q_1,0)\sqrt{|S''(q_1)|S''(q_2)}} \exp\left\{N\left[S(q_1)-S(q_2)\right]+S_1(q_1)-S_1(q_2)\right\}$$

coincides with Escudero and Kamenev (2009) who considered escape from one metastable state to another

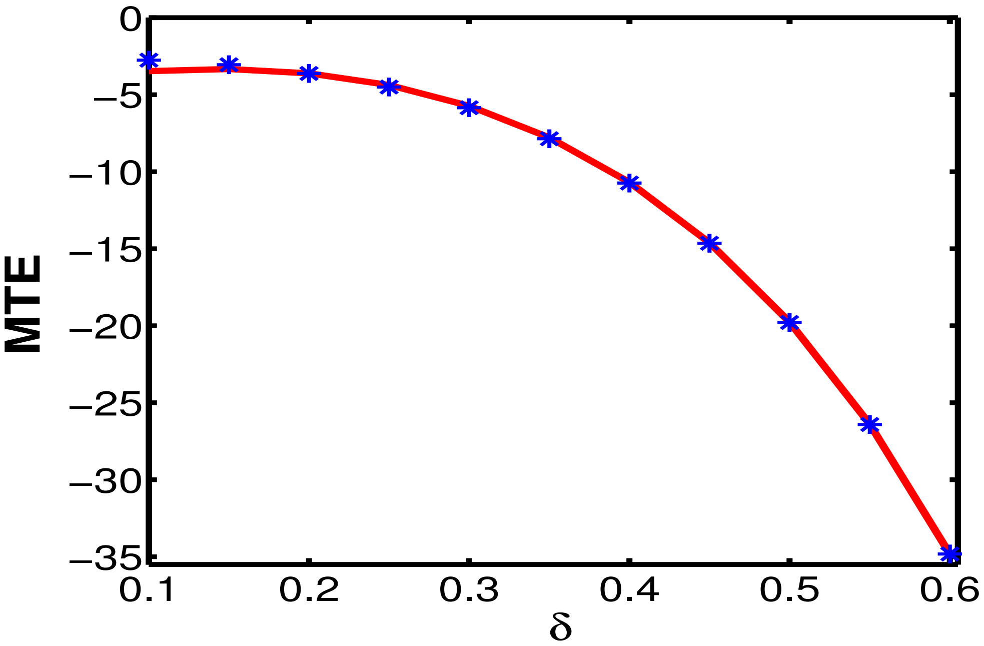
$$A \rightarrow \varnothing$$

$$2A \rightarrow 3A$$

$$3A \rightarrow 2A$$

MTE as function of **δ**

analytics vs. numerics



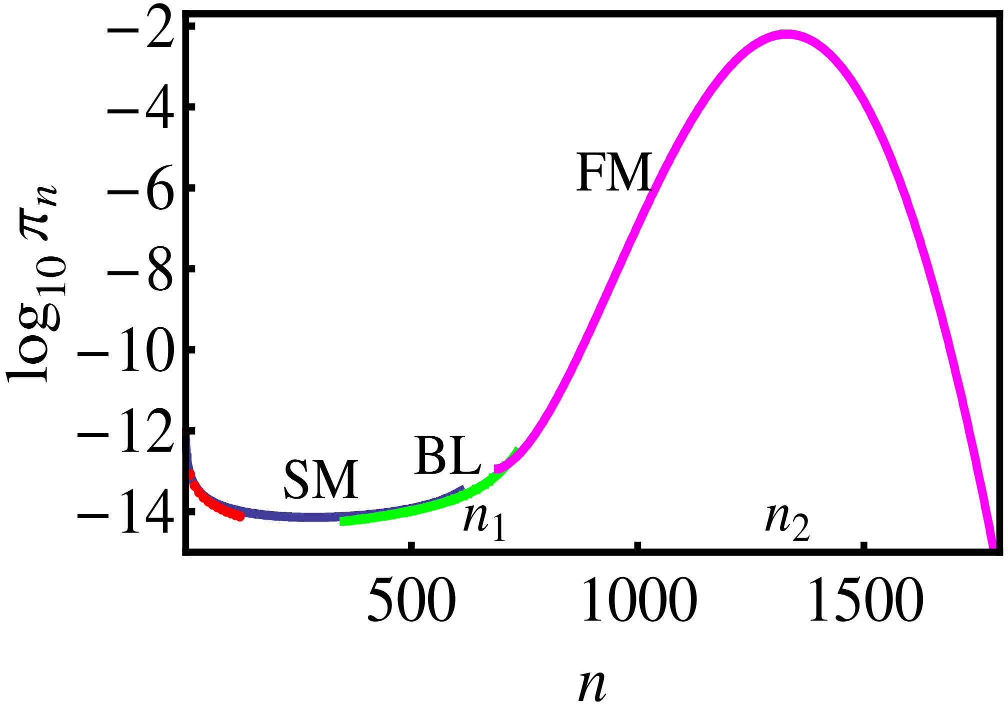
Quasi-stationary distribution

fast mode

boundary layer

slow mode

recursion



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reason for coincidence: decay determined by constant-current (slow-mode) solution

Multiple species

Example: epidemic fadeout Kamenev and M 2008

SI model with demography

Event	Type of transition	Rate
Infection	$S \rightarrow S-1, I \rightarrow I+1$	$(\beta/N)SI$
Renewal of susceptible	$S \rightarrow S+1$	μN
Death of susceptible	$S \rightarrow S-1$	μS
Death of infected	$I \rightarrow I-1$	$\mu_I I$

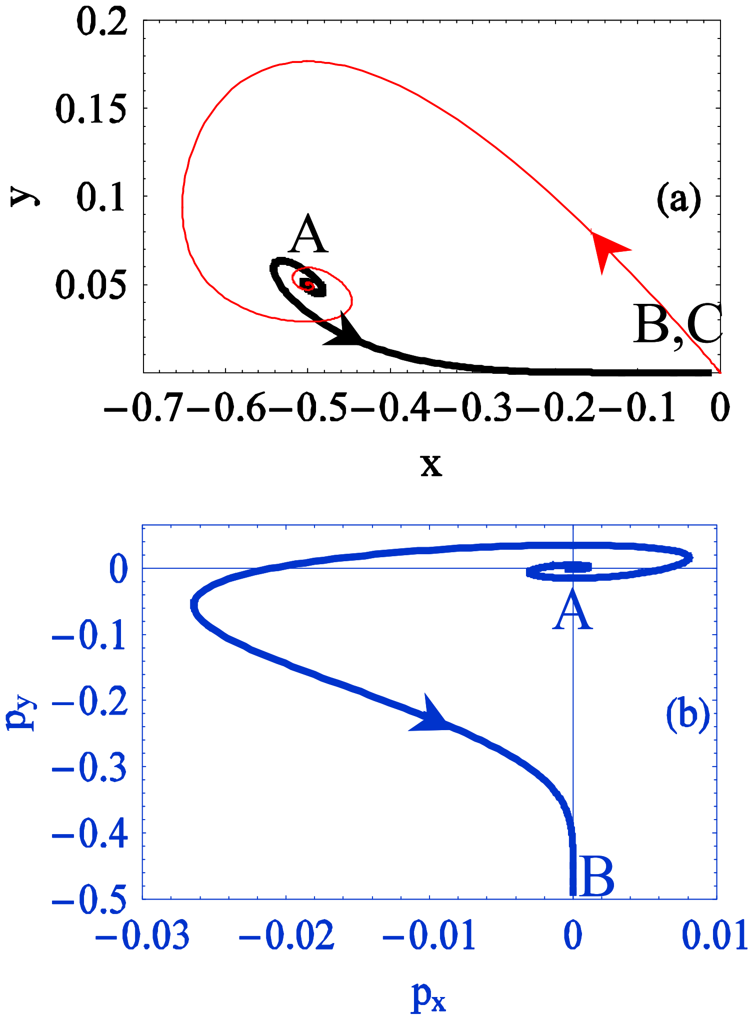
Rate equations

$$\dot{S}(t) = \mu N - \mu S - \frac{\beta}{N} SI,$$

$$\dot{I}(t) = \frac{\beta}{N} SI - \mu_I I.$$

two degrees of freedom

Stochastic model, however, predicts *extinction* of disease



disease dies out along
"the optimal path"
and may show oscillations in time

black lines: projections of
4-dimensional optimal path

"classical" action along the optimal path yields (up to pre-exponent) the logarithm of the Mean Time to Extinction of disease

Population extinction in a fluctuating environment

Example: a symmetrized losgistic model

$$\dot{P}_n(t) = \lambda_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n) P_n(t) + \lambda_{n+1} P_{n+1}(t)$$

$$\lambda_n = (n/2)(\mu + r - an), \quad \mu_n = (n/2)(\mu - r + an)$$

Environmental noise modulates the birth and death rates:

$r \rightarrow r - \xi(t)$

The noise $\xi(t)$ is red: gaussian and positively correlated

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t) \xi(t') \rangle = v \exp(-|t - t'|/t_c)$$

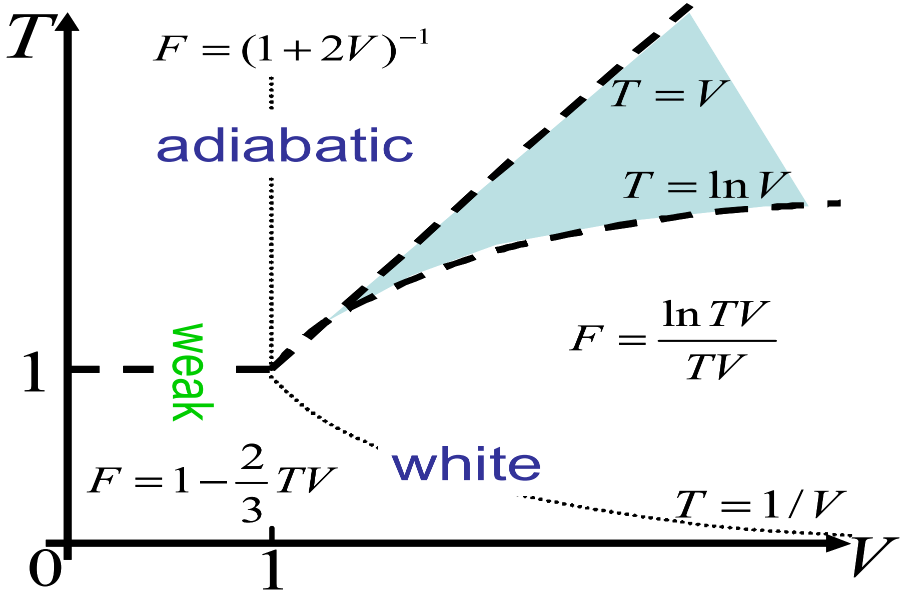
Interplay of two noises, demographic and environmental, is a hot topic in population biology and ecology. The role of noise *color* is under debate

Kamenev, Meerson and Shklovskii (PRL 2008)
Media coverage: physorg.com

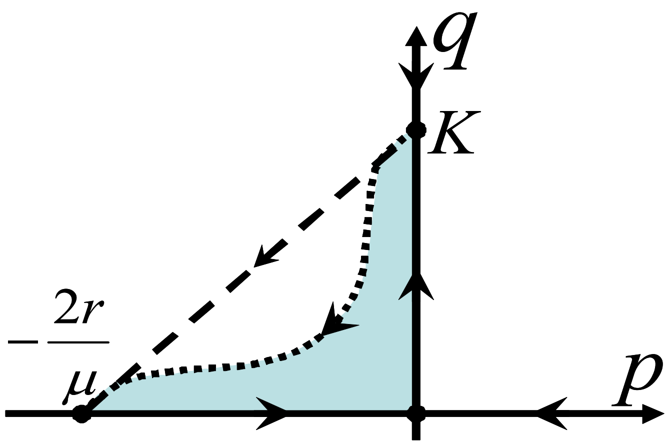
Population extinction in a fluctuating environment

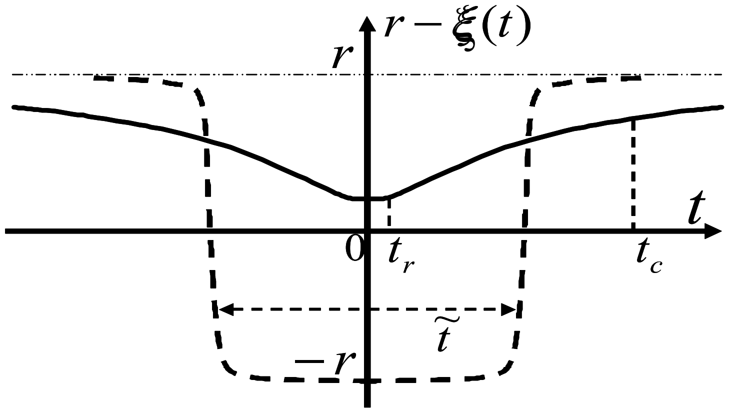
V: rescaled variance of noise
T: rescaled correlation time of noise

- Even weak noise can cause an *exponential* reduction of the MTE

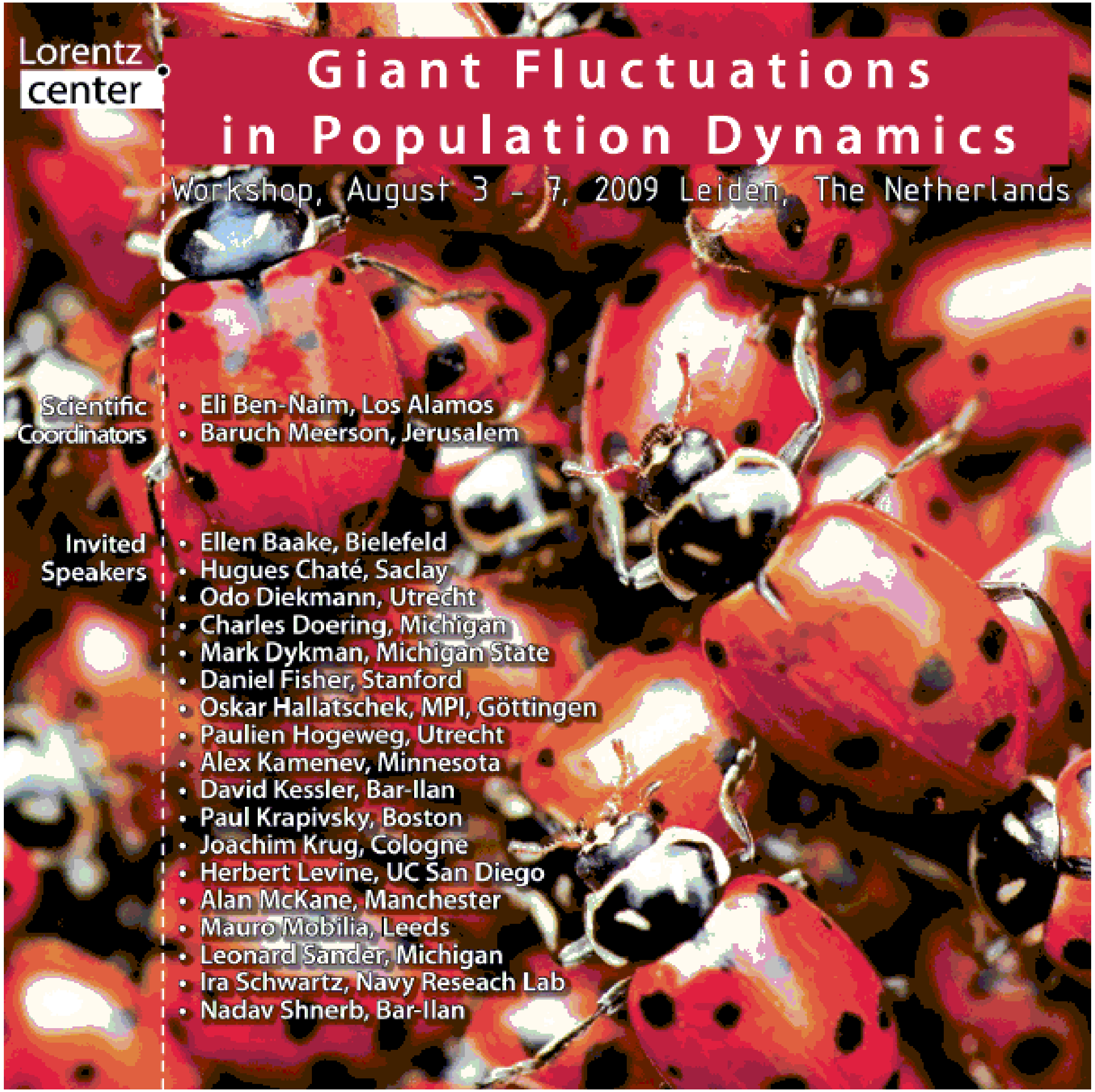


Population extinction in a fluctuating environment





- WKB yields *optimal path to extinction*, along with *optimal realization* of environmental noise.
- *Correlations* of environmental noise quicken extinction.
- The population-size dependence of the MTE changes from *exponential* without noise to a *power law* for strong short-correlated noise and to (almost) *no dependence* for long-correlated noise.



Lorentz center

Giant Fluctuations in Population Dynamics

Workshop, August 3 - 7, 2009 Leiden, The Netherlands

Scientific Coordinators

- Eli Ben-Naim, Los Alamos
- Baruch Meerson, Jerusalem

Invited Speakers

- Ellen Baake, Bielefeld
- Hugues Chaté, Saclay
- Odo Diekmann, Utrecht
- Charles Doering, Michigan
- Mark Dykman, Michigan State
- Daniel Fisher, Stanford
- Oskar Hallatschek, MPI, Göttingen
- Paulien Hogeweg, Utrecht
- Alex Kamenev, Minnesota
- David Kessler, Bar-Ilan
- Paul Krapivsky, Boston
- Joachim Krug, Cologne
- Herbert Levine, UC San Diego
- Alan McKane, Manchester
- Mauro Mobilia, Leeds
- Leonard Sander, Michigan
- Ira Schwartz, Navy Reseach Lab
- Nadav Shnerb, Bar-Ilan

Conclusions

- Two generic scenarios of extinction of metastable populations in zero dimensions
- WKB approximation (in conjunction with recursive solution and/or solution of FP eqn.) yields accurate estimates of MTE and QSD for a broad class of single-species stochastic populations
- Few species, fluctuating environment: WKB gives valuable insight: "optimal path" to extinction. Analytical results demand additional small parameters.

Thank you!