CURRENT FLUCTUATIONS IN DIFFUSIVE SYSTEMS

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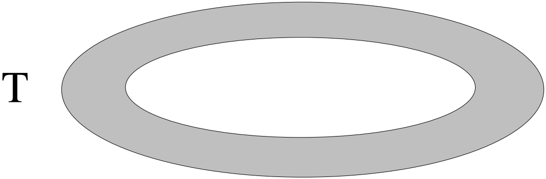
A. Gerschenfeld

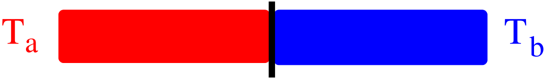
Haifa-Rehovot June 2009

Current fluctuations in non-equilibrium steady states

Universal fluctuations on a ring



Non-equilibrium initial condition



Current fluctuations in non-equilibrium steady states

B. Douçot, P.-E. Roche

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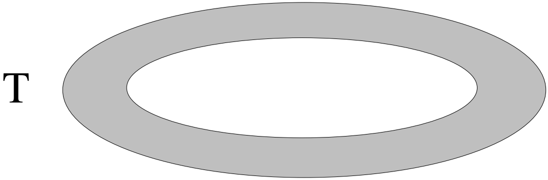
Universal fluctuations on a ring



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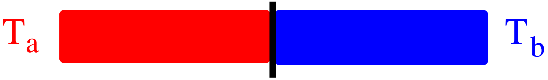
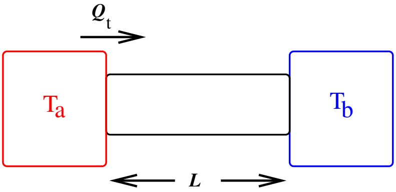


A. Gerschenfeld

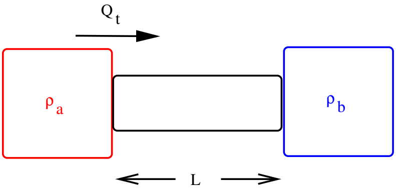
NON EQUILIBRIUM STEADY STATE

HEAT



Fourier's law

PARTICLES

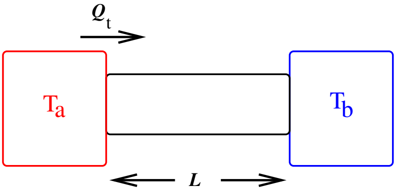


Fick's law

$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(T_a, T_b)}{L}$$

NON EQUILIBRIUM STEADY STATE

HEAT



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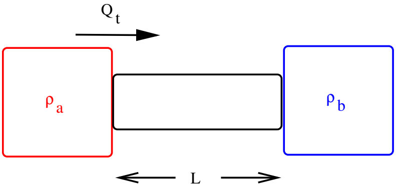


Fick's law

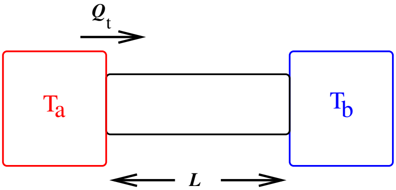
$$\frac{\langle Q_t \rangle}{t} \sim \frac{D(T_a, T_b)}{L}$$

Fluctuation Theorem

$$\log \frac{P(Q_t)}{P(-Q_t)} = (A(T_b) - A(T_a))Q_t$$

NON EQUILIBRIUM STEADY STATE

HEAT



Fourier's law

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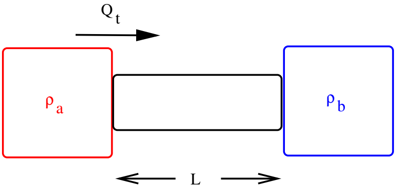

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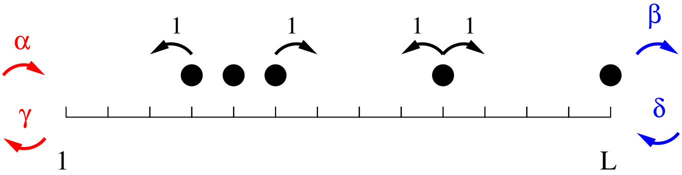
This talk

$$P(Q)?$$

EXCLUSION PROCESSES

SSEP (Symmetric simple exclusion process)



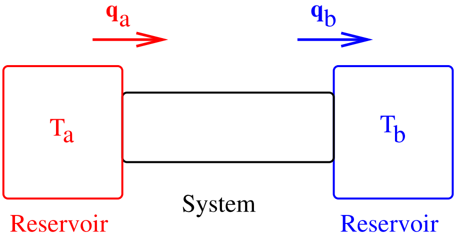
$$\rho_a = \frac{\alpha}{\alpha + \gamma} \ , \qquad \rho_b = \frac{\delta}{\beta + \delta}$$

Detailed balance $T_a = T_b = T$

$$\boxed{W(\mathcal{C}',\mathcal{C})e^{-\frac{E(\mathcal{C})}{kT}} = W(\mathcal{C},\mathcal{C}')e^{-\frac{E(\mathcal{C}')}{kT}}}$$

$$q = E(\mathcal{C}') - E(\mathcal{C}) \qquad \boxed{W_q(\mathcal{C}',\mathcal{C}) = W_{-q}(\mathcal{C},\mathcal{C}')e^{-\frac{q}{kT}}}$$

Generalized detailed balance $T_a \neq T_b$



q_a

q_b

T_a

T_b

System

Reservoir

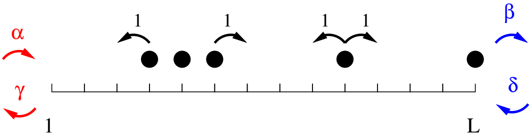
Reservoir

$$q_a - q_b = E(\mathcal{C}') - E(\mathcal{C})$$

$$\boxed{W_{q_a,q_b}(\mathcal{C}',\mathcal{C}) = W_{-q_a,-q_b}(\mathcal{C},\mathcal{C}')e^{-\frac{q_a}{kT_a}+\frac{q_b}{kT_b}}}$$

TWO APPROACHES

SSEP (Symmetric simple exclusion process)



Microscopic

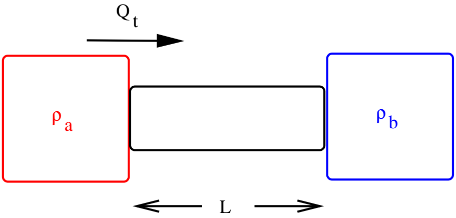
Bethe ansatz, Perturbation theory, Computer,...

Macroscopic

Rescaled density ρ and current j (with $i = Lx$, $t = L^2\tau$)

$$\boxed{\mathrm{Pro}(\{\rho(x,\tau), j(x,\tau)\}) \sim \exp\left[-L\int_0^{T/L^2} dt \int_0^1 dx\, \frac{[j+\rho']^2}{4\rho(1-\rho)}\right]}$$

GENERATING FUNCTION OF THE CURRENT



$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

The evolution

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} W(\mathcal{C},\mathcal{C}')P(\mathcal{C}') - W(\mathcal{C}',\mathcal{C})\mathcal{P}(\mathcal{C})$$

One can decompose

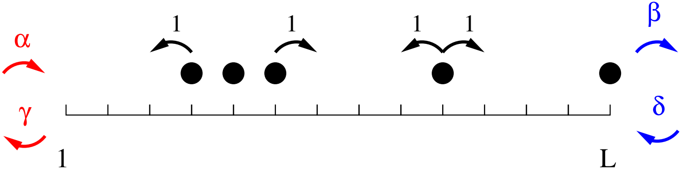
$$W(\mathcal{C},\mathcal{C}') = W_1(\mathcal{C},\mathcal{C}') + W_0(\mathcal{C},\mathcal{C}') + W_{-1}(\mathcal{C},\mathcal{C}')$$

$W_q(\mathcal{C},\mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} W_1 + W_0 + e^{-\lambda} W_{-1}}$$

EXCLUSION PROCESSES

SSEP (Symmetric simple exclusion process)



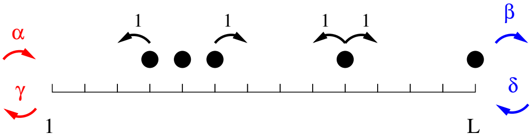
$$\rho_a = \frac{\alpha}{\alpha+\gamma}\ , \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

Steady state known

D. Domany Mukamel 1992
D. Evans Hakim Pasquier 1993
Schütz, Domany 1933

SSEP (Symmetric simple exclusion process)

D. Douçot Roche 2004

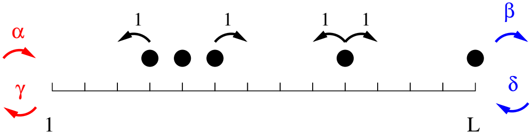


$$\rho_a = \frac{\alpha}{\alpha+\gamma}\,, \qquad \rho_b = \frac{\delta}{\beta+\delta}$$

$$\lim_{t\to\infty} \frac{\langle Q(t)\rangle}{t} \simeq \frac{1}{L}[\rho_a - \rho_b] \qquad \text{Fick's law}$$

$$\lim_{t\to\infty} \frac{\langle Q^2(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a\rho_b + \rho_b^2)}{3}\right]$$

$$\lim_{t\to\infty} \frac{\langle Q^3(t)\rangle_c}{t} \simeq \frac{1}{L}(\rho_a - \rho_b)\left[1 - 2(\rho_a+\rho_b) + \frac{16\rho_a^2 + 28\rho_a\rho_b + 16\rho_b^2}{15}\right]$$



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D. Douçot Roche 2004

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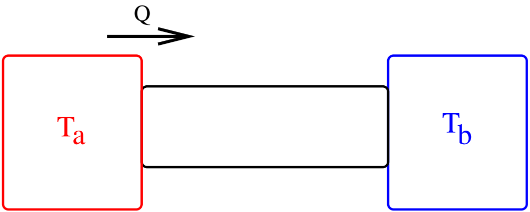
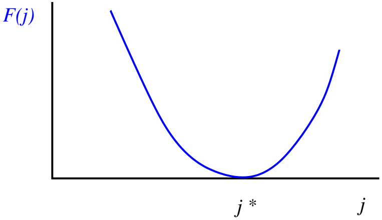
$$\lim_{t\to\infty} \frac{\langle Q^4(t)\rangle_c}{t} \simeq \frac{1}{L}\left[\rho_a + \rho_b - \frac{2(7\rho_a^2 + \rho_a\rho_b + 7\rho_b^2)}{3}\right.$$
$$\left. + \frac{32\rho_a^3 + 8\rho_a^2\rho_b + 8\rho_a\rho_b^2 + 32\rho_b^3}{5} - \frac{96\rho_a^4 + 64\rho_a^3\rho_b - 40\rho_a^2\rho_b^2 + 64\rho_a\rho_b^3 + 96\rho_b^4}{35}\right]$$

CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



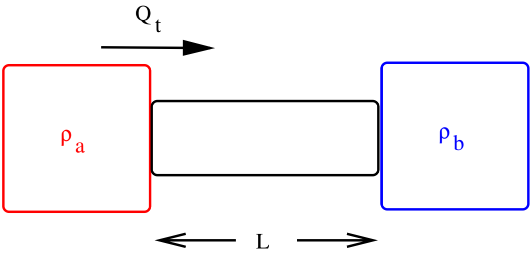
Q_t Energy transferred during time t

$$\boxed{\mathrm{Pro}\left(\frac{Q_t}{t}=j\right)\sim\exp[-t\;F(j)]}$$



$\boxed{\text{Expansion of } F(j) \text{ near } j^* \text{ gives all cumulants of } Q_t}$

DIFFUSIVE SYSTEM



Bodineau D. 2004

One assumes

- For $\rho_a - \rho_b$ small: $\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$
- $\rho_a = \rho_b = \rho$: $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

One can then calculate

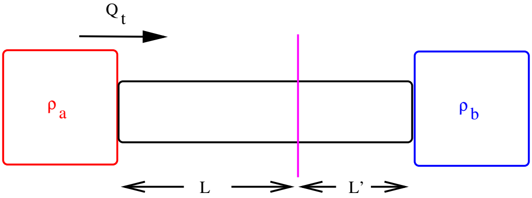
$$\boxed{\text{All cumulants of } Q_t \text{ for arbitrary } \rho_a \text{ and } \rho_b}$$

SSEP: $D = 1$ and $\sigma = 2\rho(1-\rho)$

ADDITIVITY PRINCIPLE

$$\boxed{\mathrm{Pro}\left(\frac{Q_t}{t} = j, \rho_a, \rho_b\right) \sim \exp[-t\ F_{L+L'}(j, \rho_a, \rho_b)]}$$

Bodineau D. 2004



Additivity

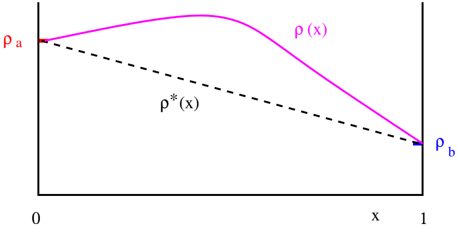
$$P_{L+L'}(Q, \rho_a, \rho_b) \sim \max_{\rho} [P_L(Q, \rho_a, \rho)\ P_{L'}(Q, \rho, \rho_b)]$$

$$\boxed{F_{L+L'}(j, \rho_a, \rho_b) = \min_{\rho} [F_L(j, \rho_a, \rho)\ +\ F_{L'}(j, \rho, \rho_b)]}$$

VARIATIONAL PRINCIPLE

Bodineau D. 2004

$$\boxed{F_L(j,\rho_a,\rho_b) = \frac{1}{L}\min_{\rho(x)} \int_0^1 dx\, \frac{[j\ L\ +\rho'(x)\ D(\rho(x))]^2}{2\sigma(\rho(x))}}$$



Satisfies the fluctuation theorem

$$F(j)-F(-j) = j\int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)}d\rho$$

Gallavotti Cohen 1995
Evans Searls 1994
.... Kurchan 1998
Lebowitz Spohn 1999

CONSEQUENCES

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}}$$

$$\mu(\lambda, \rho_a, \rho_b) = -\frac{K}{L}\left[\int_{\rho_b}^{\rho_a} \frac{D(\rho)\ d\rho}{\sqrt{1+2K\sigma(\rho)}}\right]^2$$

$$\lambda = \int_{\rho_b}^{\rho_a} d\rho \frac{D(\rho)}{\sigma(\rho)}\left[\frac{1}{\sqrt{1+2K\sigma(\rho)}} - 1\right]$$

Non universal cumulants

$$\frac{\langle Q_t \rangle_c}{t} = \frac{1}{L}\mathcal{I}_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{L}\ \frac{\mathcal{I}_2}{\mathcal{I}_1}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{L}\ \frac{3\ (\ \mathcal{I}_3\ \mathcal{I}_1 - \mathcal{I}_2^2)}{\mathcal{I}_1^3}$$

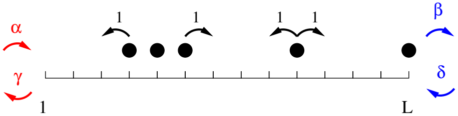
$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L}\ \frac{3\ (\ 5\ \mathcal{I}_4\ \mathcal{I}_1^2 - 14\ \mathcal{I}_1\mathcal{I}_2\ \mathcal{I}_3 +\ 9\ \mathcal{I}_2^3)}{\mathcal{I}_1^5}$$

where

$$\mathcal{I}_n = \int_{\rho_b}^{\rho_a} D(\rho)\sigma(\rho)^{n-1} d\rho$$

For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$

CURRENT FLUCTUATIONS IN THE SSEP



$$\rho_a = \frac{\alpha}{\alpha + \gamma} \qquad\qquad \rho_b = \frac{\delta}{\beta + \delta}$$

$$\boxed{\left\langle e^{Q_t} \right\rangle \sim e^{t\ \mu(\lambda)}}$$

For large L and λ small

$$\boxed{\mu(\lambda, \alpha, \gamma, \beta, \delta) = \frac{1}{L} R(\omega)}$$

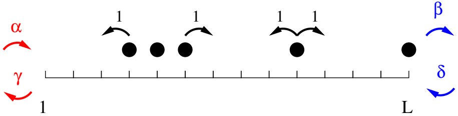
where

$$\omega = 1 - [1 - (e^{\lambda} - 1)\rho_a][1 - (1 - e^{-\lambda})\rho_b]$$

Result:

$$\boxed{R(\omega) = \left[\log(\sqrt{1+\omega} + \sqrt{\omega})\ \right]^2}$$

CURRENT FLUCTUATIONS IN THE SSEP



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$$\boxed{\langle e^{Q_t} \rangle \sim e^{t\ \mu(\lambda)}}$$

For large L and λ small $$\boxed{\mu(\lambda, \alpha, \gamma, \beta, \delta) = \frac{1}{L} R(\omega)}$$

where $$\omega = 1 - [1-(e^{\lambda}-1)\rho_a][1-(1-e^{-\lambda})\rho_b]$$

Result: $$\boxed{R(\omega) = [\log(\sqrt{1+\omega} + \sqrt{\omega})\ ]^2}$$

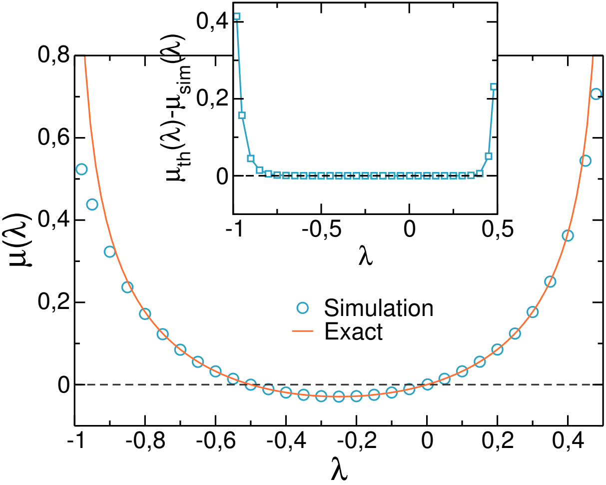
Same as the universal statistics of transport in disordered conductors

Lee, Levitov, Yakovets 1995

The Kipnis Marchioro Presutti model

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\ \mu(\lambda)}}$$

Hurtado Garrido 2008



TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$\boxed{F(j) = \frac{1}{L} \lim_{T \to \infty} \min_{\rho(x,t), j(x,t)} \frac{1}{T} \int_0^T dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$ and

$$j\ T = \int_0^T j_t(x) dt$$

- Sufficient condition for the optimal profile to be time independent
- Dynamical phase transition

Bodineau D. 2005-2007
Prohlac Mallick 2009

$$\boxed{\text{the optimal } \rho_t(x) \text{ starts to become time dependent}}$$

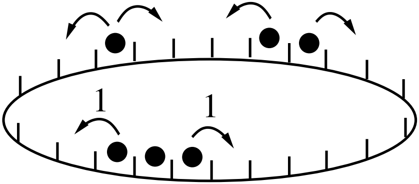
SSEP ON A RING

Appert D Lecomte Van Wijland 2008

N particles

L sites

$$\rho = \frac{N}{L}$$



Q_t flux through a bond during time t

CUMULANTS OF THE CURRENT FOR THE SSEP ON A RING

N particles and L sites $\qquad \sigma(\rho) = 2\rho(1-\rho) = \frac{2N(L-N)}{L^2}$

$$\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L-1}$$

$$\frac{\langle Q^4\rangle_c}{t} = \frac{\sigma^2}{2(L-1)^2}$$

$$\frac{\langle Q^6\rangle_c}{t} = -\frac{(L^2-L+2)\sigma^3-2(L-1)\sigma^2}{4(L-1)^3(L-2)}$$

$$\frac{\langle Q^8\rangle_c}{t} = \frac{(10L^4-2L^3+27L^2-15L+18)\sigma^4-4(L-1)(11L^2-L+12)\sigma^3+48(L-1)^2\sigma^2}{24(L-1)^4(L-2)(L-3)}$$

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$

Gaussian + Fick's law

$$\boxed{\frac{\langle Q^{2n}\rangle_c}{t} \sim \frac{\sigma^n}{L^2}}$$

for $n \geq 2$

UNIVERSAL CUMULANTS OF THE CURRENT

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$

Gaussian

$$\boxed{\frac{\langle Q^4\rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6\rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8\rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}}$$

Universal

UNIVERSAL CUMULANTS OF THE CURRENT

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$

Gaussian

$$\boxed{\frac{\langle Q^4\rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6\rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8\rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}}$$

Universal

$$\boxed{\langle e^{\lambda Q_t}\rangle \sim e^{t\,\mu(\lambda)}} \quad \text{with} \quad \boxed{\mu(\lambda) - \frac{\lambda^2}{2}\frac{\langle Q^2\rangle_c}{t} = \frac{1}{L^2}\,\mathcal{F}\left(-\frac{\sigma\lambda^2}{4}\right)}$$

$$\mathcal{F}(u) = -4\sum_{n\geq 1}\left[n\pi\sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u\right] = \frac{1}{3}u^2 + \frac{1}{45}u^3 + \frac{1}{378}u^4 + \ldots$$

$$\boxed{\mathcal{F} \ \text{universal}}$$

Singularity as $u \to \frac{\pi^2}{2}$

BETHE ANSATZ

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

The evolution

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')P(\mathcal{C}') - M(\mathcal{C}',\mathcal{C})\mathcal{P}(\mathcal{C})$$

One can decompose

$$M(\mathcal{C},\mathcal{C}') = M_1(\mathcal{C},\mathcal{C}') + M_0(\mathcal{C},\mathcal{C}') + M_{-1}(\mathcal{C},\mathcal{C}')$$

$M_q(\mathcal{C},\mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} M_1 + M_0 + e^{-\lambda} M_{-1}}$$

BETHE ANSATZ EQUATIONS

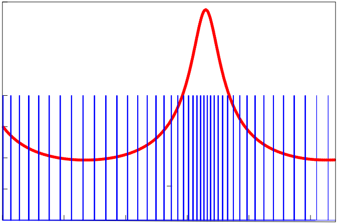
Define $s = \lambda / L$

$$\boxed{z_i^L = \prod_{\substack{j=1 \\ j \neq i}}^{N} \left[- \frac{e^s - 2z_i + e^{-s} z_i z_j}{e^s - 2z_j + e^{-s} z_i z_j} \right] ,}$$

Then

$$\mu(\lambda) = \sum_i^N z_i e^{-s} + \frac{1}{z_i} e^s - 2$$

- The z_i accumulate on a line
- Universality comes from the discrete nature of the z_i



FLUCTUATING HYDRODYNAMICS

Gaussian expansion of the **macroscopic fluctuation theory** around a constant current and a flat profile.

$$\rho(x,t) = \rho + \sum_{k,\omega} k[a_{k,\omega} e^{i\omega\tau + ikx} + a^*_{k,\omega} e^{-i\omega t - ikx}]$$

$$j = j_0 - \omega[a_{k,\omega} e^{i\omega\tau + ikx} + a^*_{k,\omega} e^{-i\omega t - ikx}] \ .$$

Gaussian fluctuations

$$\mathrm{Pro}(Q_t = j_0 t, \{a_{k,\omega}\}) \sim$$
$$\exp\left[-\frac{j_0^2}{2\sigma}\frac{t}{L} - \frac{t}{L}\sum_{\omega,k} |a_{k,\omega}|^2 \left(\frac{(\sigma\omega + j_0\sigma' k)^2}{\sigma^3} + \frac{D^2 k^4}{\sigma} - \frac{j_0^2 \sigma'' k^2}{2\sigma^2} \right) \right]$$

- Integrate over the fluctuations
- Sum over the discrete modes k

RESULTS FOR A GENERAL DIFFUSIVE SYSTEM

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}}$$

$$\mu(\lambda) - \frac{\lambda^2 \langle Q^2 \rangle}{2t} = \frac{1}{L^2} D \mathcal{F}\left(\frac{\sigma \sigma''}{16 D^2} \lambda^2 \right)$$

$$\mathcal{F}(u) = -4 \sum_{n \geq 1} \left[n\pi \sqrt{n^2\pi^2 - 2u} - n^2\pi^2 + u \right] = \frac{1}{3} u^2 + \frac{1}{45} u^3 + \frac{1}{378} u^4 + \dots$$

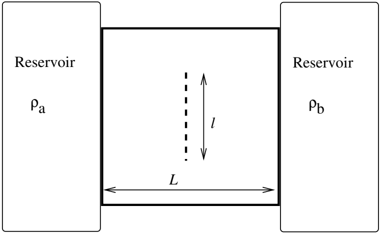
- Phase transition as $u \to \pi^2/2$
- For $n \geq 2$

$$\boxed{\frac{\langle Q^{2n} \rangle_c}{t} \sim \frac{1}{L^2} \; \frac{(2n)!\, B_{2n-2}}{n!\, (n-1)!} \; D \left(\frac{\sigma \sigma''}{8 D^2} \right)^n}$$

B_n Bernoulli numbers

DIFFUSIVE SYSTEM IN TWO DIMENSION

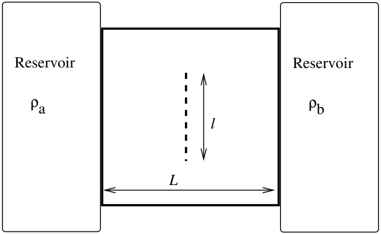
Bodineau D. Lebowitz 2008



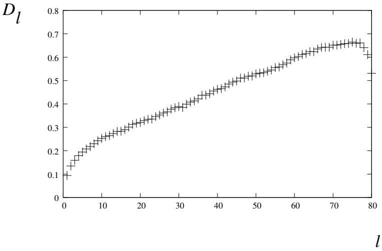
$$\frac{\langle Q^2 \rangle}{t} = D_l$$

Macroscopic fluctuation theory: $D_l = \infty$

SSEP IN TWO DIMENSION

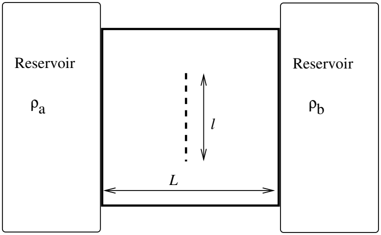


$$\frac{\langle Q^2\rangle}{t} = D_l$$

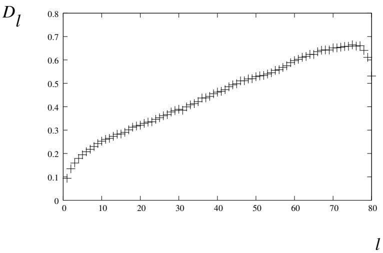


$$\rho_a = \rho_b$$

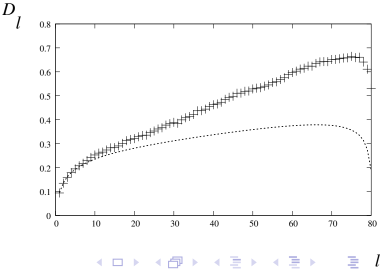
SSEP IN TWO DIMENSION



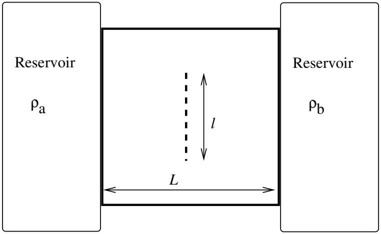
$$\frac{\langle Q^2\rangle}{t} = D_l$$

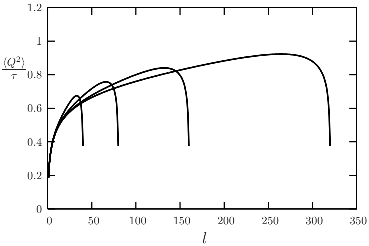


$$\lim_{t\to\infty}\frac{\langle Q^2\rangle}{t} = D_l$$

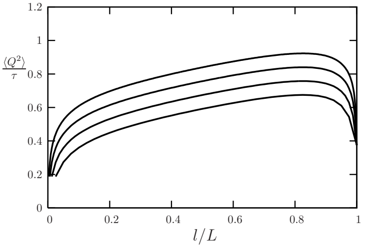


VORTICES IN TWO DIMENSION





$$\frac{\langle Q^2\rangle}{t} = D_l$$

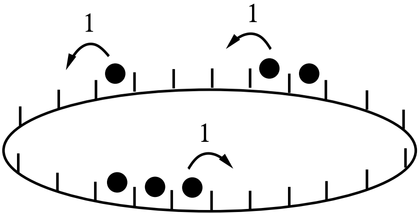


- For $\frac{l}{L} \neq 0, 1$

$$\boxed{D_l \simeq A \log L + F\left(\frac{l}{L}\right)}$$

- For $l \ll L$ $\qquad D_l \simeq G\left(l\right)$

TASEP (totally asymmetric exclusion process)- PHASE TRANSITION



N particles $\quad L$ sites

$$\rho = \frac{N}{L}$$

$$\langle Q \rangle = t\rho(1-\rho)$$

D. Lebowitz 1998; Appert D. 1999

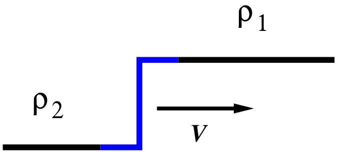
$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}}$$

For $\lambda < 0$

$$\boxed{\mu(\lambda) = -\frac{(1-e^{\lambda\rho})(1-e^{\lambda(1-\rho)})}{1-e^{\lambda}}}$$

TASEP:

shocks



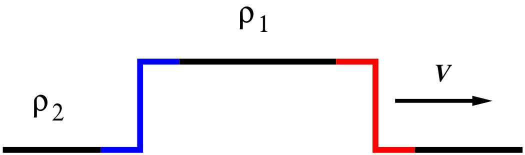
$$\boxed{v = 1 - \rho_1 - \rho_2}$$

Current $j(\rho) = \rho(1-\rho) \quad \Rightarrow \quad v = \frac{j(\rho_1) - j(\rho_2)}{\rho_1 - \rho_2}$

TASEP:

shocks and anti shocks on the line

Jensen Varadhan 2000



$$v = 1 - \rho_1 - \rho_2$$

$$\mathrm{Pro}(\text{anti shock}) = \exp[-t\ \Phi(\rho_1, \rho_2)]$$

where

$$\Phi(\rho_1, \rho_2) = \rho_2 - \rho_1 - \rho_1\rho_2 \log \frac{\rho_2}{\rho_1} - (1-\rho_1)(1-\rho_2) \log \frac{1-\rho_2}{1-\rho_1}$$

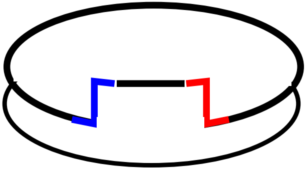
LARGE DEVIATION FUNCTION

Bodineau D. 2005

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}}$$

For $\lambda < 0$

$$\boxed{\mu(\lambda) = -\frac{(1-e^{\lambda\rho})(1-e^{\lambda(1-\rho)})}{1-e^{\lambda}}}$$

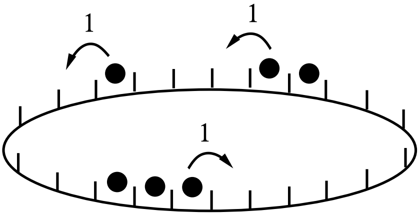


One can recover this expression by

$$\boxed{\mu(\lambda) = \max_{\rho_1,\rho_2} \{[\lambda[y\rho_1(1-\rho_1) + (1-y)\rho_2(1-\rho_2)] - \Phi(\rho_1,\rho_2)\}}$$

where $\rho = y\rho_1 + (1-y)\rho_2$

TASEP (totally asymmetric exclusion process)- PHASE TRANSITION



N particles $\quad L$ sites

$$\rho = \frac{N}{L}$$

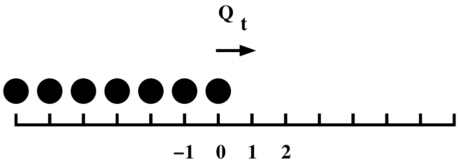
$$\langle Q \rangle = t\rho(1-\rho)$$

D. Lebowitz 1998; Appert D. 1999

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \qquad \mu(\lambda) - \frac{\lambda \langle Q \rangle}{t} = \sqrt{\frac{\rho(1-\rho)}{L^3}}\; \mathcal{G}\left(\lambda\sqrt{L\rho(1-\rho)}\right)$$

$$\boxed{\mathcal{G}(x) = -\frac{1}{\sqrt{2\pi}} \sum_{n\geq 1} z^n n^{-3/2} \quad ; \quad x = -\frac{1}{\sqrt{2\pi}} \sum_{n\geq 1} z^n n^{-5/2}}$$

STEP INITIAL CONDITION



ASEP

Schütz 98

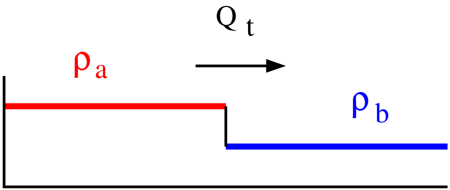
⋮

Johansson 2000

Prähofer Spohn 2000-2002

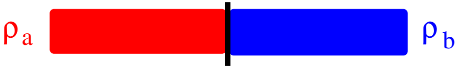
⋮

Tracy Widom 2008



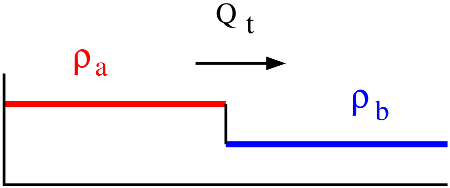
SSSEP

D Gerschenfeld 2009



$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle = ?}$$

STEP INITIAL CONDITION



SSSEP

D Gerschenfeld 2009

$$\boxed{\langle e^{\lambda Q_t} \rangle \simeq \exp[\sqrt{t}\ F(\omega)]}$$

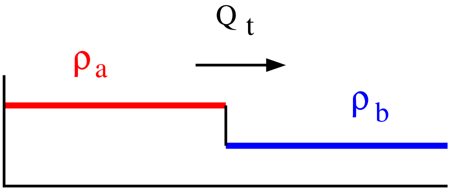
where

$$\boxed{\omega = \rho_a(e^{\lambda} - 1) + \rho_b(e^{-\lambda} - 1) + \rho_a \rho_b (e^{\lambda} - 1)(e^{-\lambda} - 1)}$$

and

$$F(\omega) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 1} \frac{(-)^{n+1}}{n^{3/2}}\ \omega^n \equiv \frac{1}{\pi} \int_{\infty}^{\infty} dk \log\left[1 + \omega e^{-k^2}\right]$$

STEP INITIAL CONDITION



SSSEP

D Gerschenfeld 2009

$$\boxed{\langle e^{\lambda Q_t} \rangle \simeq \exp[\sqrt{t}\ F(\omega)]}$$

with

$$F(\omega) = \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[1 + \omega e^{-k^2} \right]$$

For large Q_t

$$\boxed{\mathrm{Pro}(Q_t) \sim \exp[-Q_t^3/t]}$$

CONCLUSION

Universality on the ring

Macroscopic fluctuation theory versus Bethe ansatz

OPEN QUESIONS

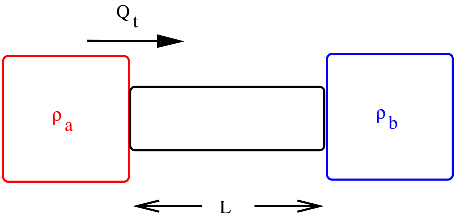
Open system: ASEP

Ring: WASEP (bias $\sim L^{-1}$) and phase transition

More than one conserved quantity

Macroscopic fluctuation theory for a non-equilibrium initial condition
A. Gerschenfeld

GENERATING FUNCTION OF THE CURRENT



$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

The evolution

$$\frac{dP(\mathcal{C})}{dt} = \sum_{\mathcal{C}'} W(\mathcal{C},\mathcal{C}')P(\mathcal{C}') - W(\mathcal{C}',\mathcal{C})\mathcal{P}(\mathcal{C})$$

One can decompose

$$W(\mathcal{C},\mathcal{C}') = W_1(\mathcal{C},\mathcal{C}') + W_0(\mathcal{C},\mathcal{C}') + W_{-1}(\mathcal{C},\mathcal{C}')$$

$W_q(\mathcal{C},\mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} W_1 + W_0 + e^{-\lambda} W_{-1}}$$

GENERATING FUNCTION OF THE CURRENT

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} W_1 + W_0 + e^{-\lambda} W_{-1}}$$

The evolution

$$\frac{dP(\mathcal{C}, Q)}{dt} = \sum_{q} \sum_{\mathcal{C}'} W_q(\mathcal{C}, \mathcal{C}') P(\mathcal{C}', Q - q) - W_q(\mathcal{C}', \mathcal{C}) P(\mathcal{C}, Q)$$

$W_q(\mathcal{C}, \mathcal{C}')$ represents a jump $\mathcal{C}' \to \mathcal{C}$ with $Q_t \to Q_t + q$

GENERATING FUNCTION OF THE CURRENT

$$\boxed{\left\langle e^{\lambda Q_t} \right\rangle \sim e^{t\,\mu(\lambda)}} \quad ?$$

$$\boxed{\mu(\lambda) = \text{largest eigenvalue of } e^{\lambda} W_1 + W_0 + e^{-\lambda} W_{-1}}$$

The evolution

$$\sum_Q e^{\lambda Q} \left(\frac{dP(\mathcal{C}, Q)}{dt} = \sum_q \sum_{\mathcal{C}'} W_q(\mathcal{C}, \mathcal{C}') P(\mathcal{C}', Q - q) - W_q(\mathcal{C}', \mathcal{C}) P(\mathcal{C}, Q) \right)$$

Define $\quad R(\mathcal{C}, \lambda) = \sum_Q P(\mathcal{C}, Q) e^{\lambda Q}$

$$\frac{dR(\mathcal{C}, \lambda)}{dt} = \sum_q \sum_{\mathcal{C}'} e^{\lambda q} W_q(\mathcal{C}, \mathcal{C}') R(\mathcal{C}', \lambda) - W_q(\mathcal{C}', \mathcal{C}) R(\mathcal{C}, \lambda)$$