Extreme Statistics in Real-space Condensation

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- A brief review on Extreme Value Statistics of i.i.d random variables

$\Longrightarrow$ three limiting distributions: GUMBEL, FRÉCHET & WEIBULL

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- Summary and Conclusions

Law of Averages: Central Limit Theorem

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- If $p(x) \sim x^{-\gamma}$ $(0 < \gamma < 3)$: $\bar{x} \Longrightarrow$ LÉVY laws

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$\Longrightarrow a_N = \ln N$, $b_N = 1$ and $F(z) = e^{-e^{-z}} \to$ GUMBEL

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Several applications $\Longrightarrow$ Climate, Finance, Oceanography, Disordered Systems (Random Energy Model of Derrida),.....

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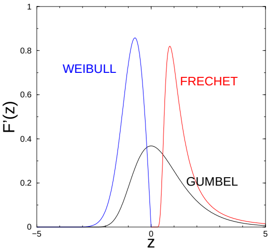

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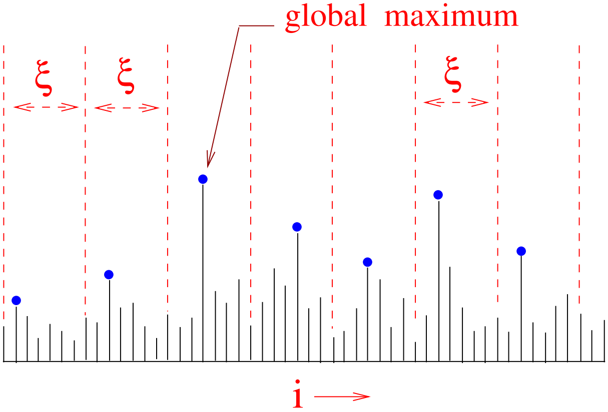

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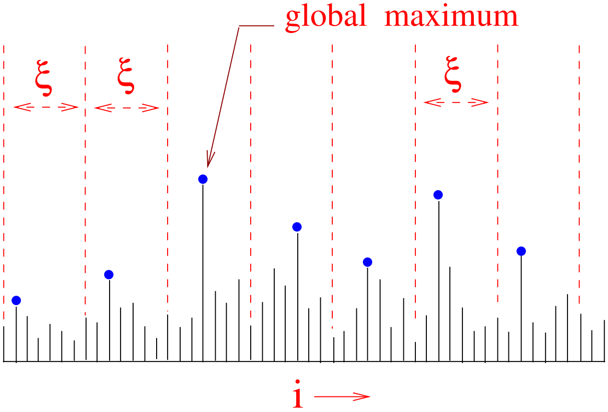

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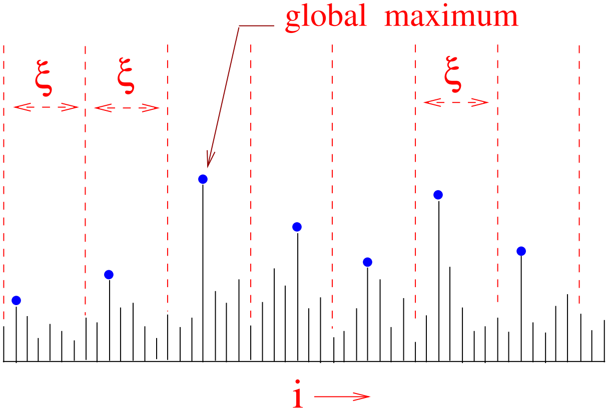

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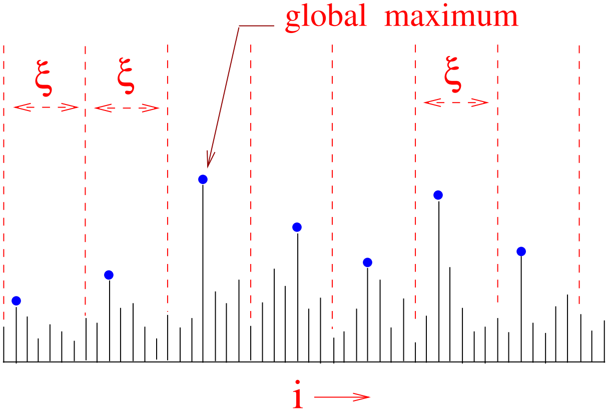

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- hierarchically correlated variables: directed polymer on a tree (Derrida & Spohn, 87)

  $\rightarrow$ distribution of the minimal energy (Dean & S.M, 2001).

Globally Constrained System: Condensation

Variables $\{x_1, x_2, \ldots, x_N\}$ $\longrightarrow$ are i.i.d but only upto a global constraint

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- For a class of $f(x)$: there is a critical value M_c

$M < M_c \Longrightarrow$ Gumbel

$M > M_c \Longrightarrow$ new universal distributions

Real-space Condensation in Mass Transport Processes

- Transport of a scalar from site to site $\longrightarrow$ mass/particles/energy

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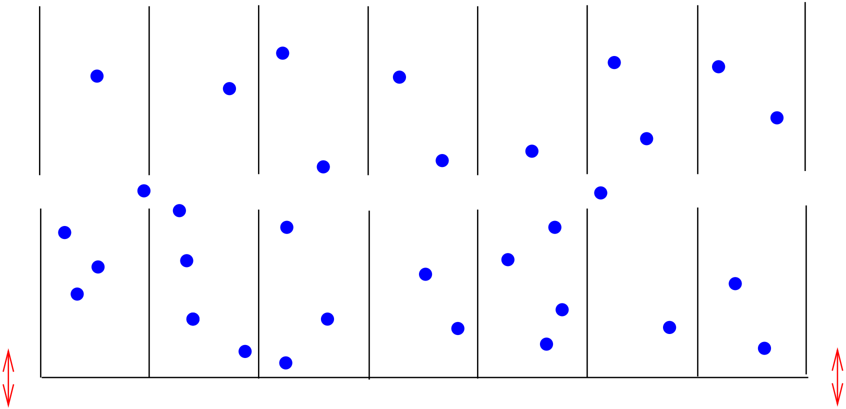
Macroscopic amount of the scalar (mass) condenses onto a single site in space

$\Longrightarrow$ Spontaneous Symmetry Breaking

Shaken Granular System

Shaken granular system: (van der Meer et. al., 2007)



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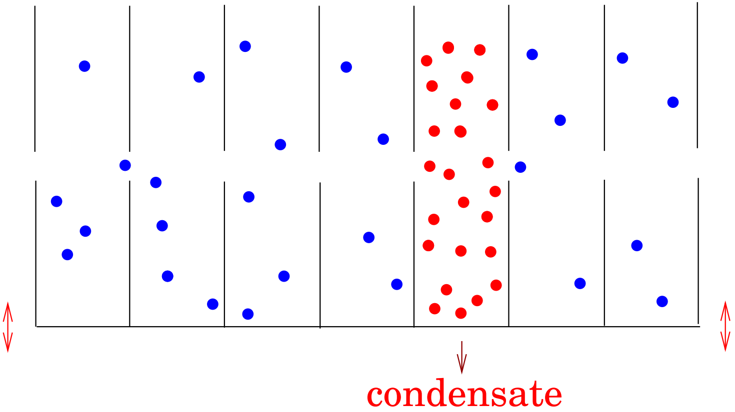

Physical Systems with Real-space Condensation:

- Traffic and Granular flow (O'Loan, Evans, Cates, '98)
- Cluster Aggregation and Fragmentation (S.M, Krishnamurthy, Barma, '98)
- Granular clustering (van der Meer et. al., 2000)
- Phase separation in driven systems (Kafri et. al., 2002).
- Socio-economic contexts: company formation, city formation, wealth condensation etc. (Burda et. al., 2002)
- Networks (Dorogovstev & Mendes, 2003,....)
- ...

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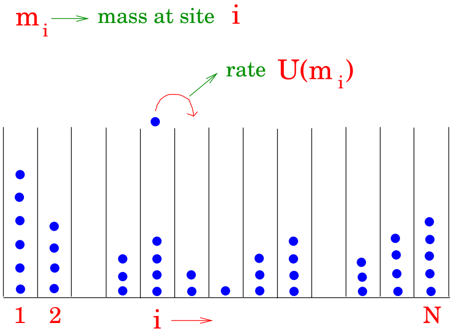

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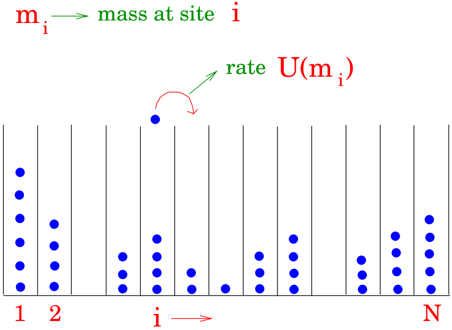

Dynamics: in time dt, each site i with mass $m_i > 0$

(i) transfers a single unit of mass to $i+1$ with prob. $U(m_i)\,dt$

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By choosing the transfer rate $U(m)$

$\Longrightarrow$ a class of $f(m)$

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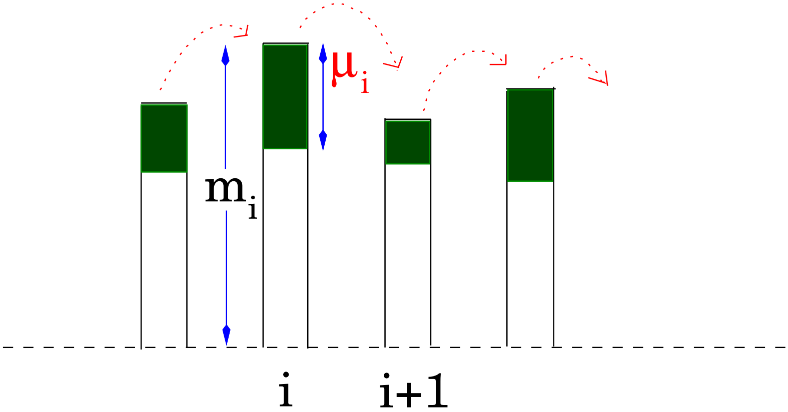
Generalized mass transport model (Evans, S.M. & Zia, 2004)



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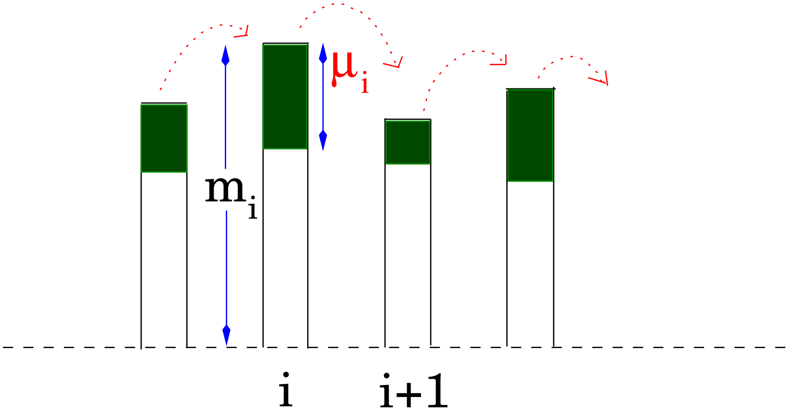

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Different choices of the kernel $\alpha(\mu|m) \Longrightarrow$ various models

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$\Longrightarrow$ In general, hard to answer !

$\rightarrow$ brief review on mass transport model (S.M., Les Houches 2008, arXiv:0904.4097)

Product Measure: Necessary and Sufficient Condition

If (and only if) the kernel $$\boxed{\alpha(\mu|m) = y(\mu)\,\frac{f(m-\mu)}{f(m)}}$$

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Generalization to parallel dynamics, arbitrary graphs ...

(Evans, S.M., Zia, 2004-2007)

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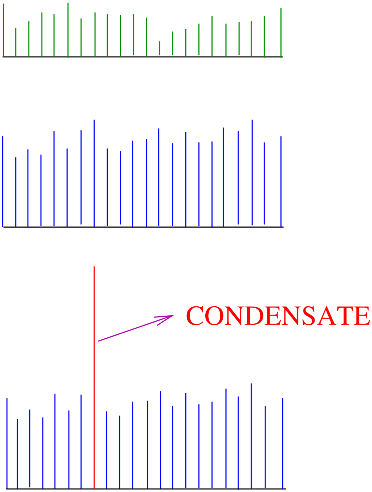
$$
\begin{aligned}
p(m) &\simeq e^{-m/m^*} && \text{for } \rho < \rho_c \\
&\simeq m^{-\gamma} && \text{for } \rho = \rho_c \\
&\simeq m^{-\gamma} + \text{"condensate"}, && \text{for } \rho > \rho_c
\end{aligned}
$$

Typical Configurations: Subcritical, Critical and Supercritical



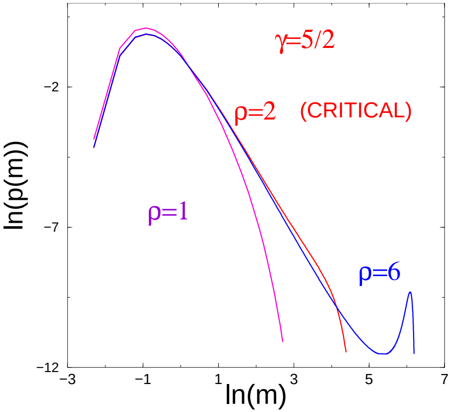
$\rho < \rho_c$

$\rho = \rho_c$

(CRITICAL)

$\rho > \rho_c$

Single Site Mass Distribution



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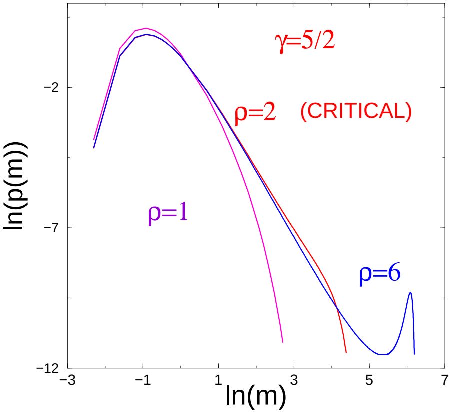

For $\rho > \rho_c$: $\int_{\text{bump}} p(m)\, dm \simeq 1/N \rightarrow$ single condensate

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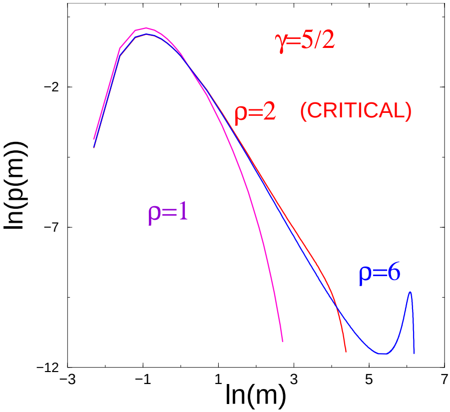

For $\rho > \rho_c$:

$$\int_{\rm bump} p(m)\, dm \simeq 1/N \rightarrow$$ single condensate

$$\int_{\rm bump} m\, p(m)\, dm \simeq (\rho - \rho_c) \rightarrow$$ extra density

Analysis of the Maximal Mass: Sub-critical Phase

$\rho < \rho_c$

- Product Measure steady state

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$$\delta\left(\sum m_i - M\right) \to e^{-s\,(m_1 + m_2 + \ldots m_N)}$$

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As ρ increases, a solution $s > 0$ exists as long as $\rho < g(0) = \rho_c$

Analysis of the Maximal Mass: Sub-critical Phase $\rho < \rho_c$

- Product Measure steady state

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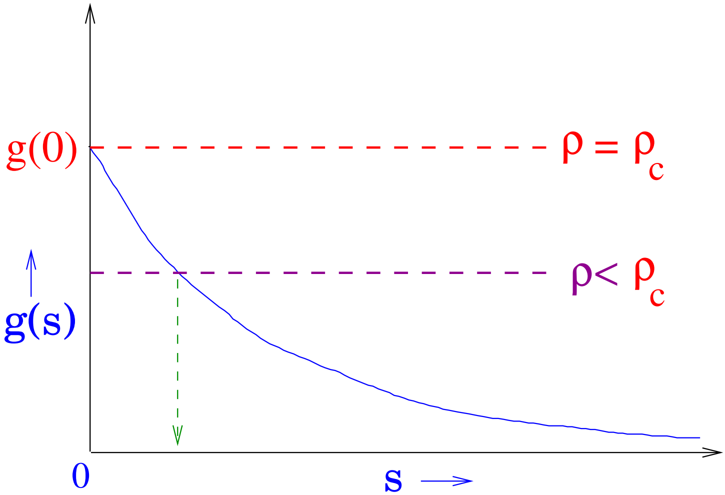
- In the sub-critical phase $\rho < \rho_c$ maximal mass distribution

Critical Density



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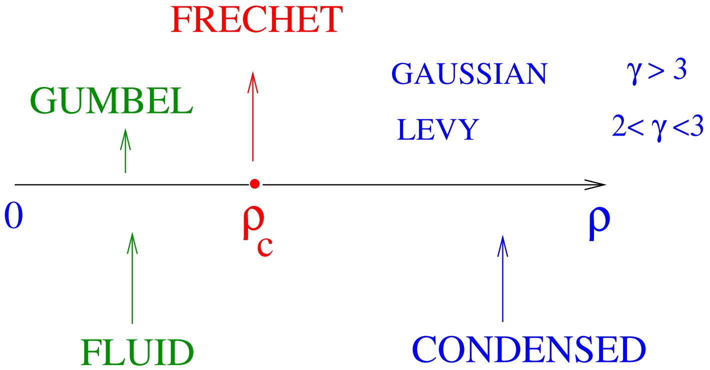
$\simeq c_1 z^{(3-\gamma)/(\gamma-2)}\, \exp\left[-c_2 z^{(\gamma-1)/(\gamma-2)}\right]$ as $z \to \infty$

Summary of Maximal Mass Distribution

EXTREME VALUE DISTRIBUTION



(Evans & S.M, 2008)

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- Extreme Value Statistics in Strongly correlated system

  $\Longrightarrow$ interesting open problems

References

Jointly with M.R. Evans and R.K.P. Zia:

- J. Phys. A: Math. Gen. **37**, L275 (2004)
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- J. Stat. Phys. **123**, 357 (2006)
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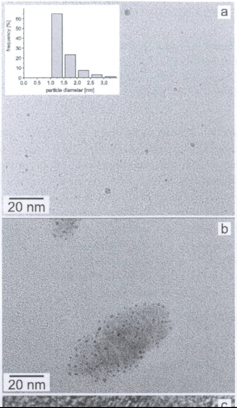
A recent brief review on real-space condensation:

S.M., Les Houches Lecture Notes (2008), arXiv: 0904.4097

Island formation: Au clusters on Carbon films



and their stability tested in a heating experiment (Sec. IV D).

In Sec. V the formation of the islands and the shape of the Au clusters on their surface is modeled with MC simulations, followed by the conclusion in Sec. VI.

II. METHODS

A. Experiment: Substrate and sample preparation

The commercial aC substrate films were produced by evaporation in a carbon arc by Arizona Carbon Foils and distributed by Plano GmbH as type S160. The films are mounted on a 200 nm mesh Cu grid for support. The film thickness is given by the manufacturer as d_{subs} $=10-12.5$ nm with a density of $\rho_{subs} \approx 2.0$ g/cm^3, corresponding to a mass of $2-2.5$ μg/cm^2. The density of the substrate is closer to that of graphite ($\rho_{graph}=2.267$ g/cm^3) than to that of diamond ($\rho_{dia}=3.515$ g/cm^3), suggesting a structure that contains regions with trivalent coordination.[23] The similarities in the electronic structure between aC and graphite are supported by electron energy loss spectroscopy (EELS) measurements,[24] which show spectra that differ significantly from those of diamond. Amorphous-carbon films produced with similar methods and of similar density have been found to be semiconducting.[23,25]